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Certain bivariate distributions and random processes connected with maxima and minima

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Certain bivariate distributions and random processes connected with maxima and minima

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Abstract: It is well-known that $[S(x)]^n$ and $[F(x)]^n$ are the survival function and the distribution function of the minimum and the maximum of n independent, identically distributed random variables, where S and F are their common survival and distribution functions, respectively. These two extreme order statistics play important role in countless applications, and are the central and well-studied objects of extreme value theory. In this work we provide stochastic representations for the quantities $[S(x)]^\alpha$ and $[F(x)]^\alpha$, where $\alpha > 0$ is no longer an integer, and construct a bivariate model with these margins. Our constructions and representations involve maxima and minima with a random number of terms. We also discuss generalizations to random process and further extensions.

AMS 2000 subject classifications:

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1. Introduction

Let $\{X_i\}$ be a sequence of independent and identically distributed (IID) random variables with cumulative distribution function (CDF) F and survival function (SF) S. The extreme order statistics connected with this sequence, the minimum

$$S_n = \bigwedge_{j=1}^{n} X_j = \min_{1\le j\le n}\{X_1, \ldots, X_n\} \tag{1.1}$$

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and the maximum

$$M_n = \bigvee_{j=1}^{n} X_j = \max_{1\le j\le n} \{X_1, \ldots, X_n\}, \tag{1.2}$$

which are crucial in countless applications, are the central and well-studied objects of extreme value theory (see, e..g., monographs [2, 3, 8, 9, 11, 16, 19, 26, 27] and extensive references therein as well as in [15]). It is well-known (see, e.g, [1] or [6]) that the SF of S_n and the CDF of M_n are given by $[S(x)]^n$ and $[F(x)]^n$, respectively, while the joint CDF of the vector (S_n, M_n) has the form

$$\mathbb{P}(S_n \le x, M_n \le y) = \begin{cases} [F(y)]^n - [F(y) - F(x)]^n & \text{for } x < y \\ [F(y)]^n & \text{for } x \ge y. \end{cases} \tag{1.3}$$

It is rather obvious that the quantities

$$[S(x)]^\alpha \text{ and } [F(x)]^\alpha \tag{1.4}$$

are genuine survival and distribution functions, respectively, for any positive value of α, although the interpretations (1.1) and (1.2) no longer apply when α is not an integer. Nevertheless, many interesting, flexible, and useful univariate families of distributions have been defined this way over the years, with earliest works connected with an exponential model going back to the first half of the nineteenth century (see, e.g., [10]). In addition to the exponential, these new classes of distributions - often referred to as *exponentiated distributions* - include those connected with gamma, Pareto, and Weibull laws (see, e.g., [12, 13, 14, 21, 22, 24] and references therein).

In this work we study a *bivariate* generalization of the joint distribution (1.3) in the same spirit, where the margins are given by (1.4) and the bivariate model reduces to (1.3) when α is an integer. Moreover, the bivariate model with a non-integer α, as well as the margins (1.4), are constructed through maxima and minima. This generalization does not arise by simply replacing the integer n by α in the bivariate CDF (1.3), as the latter fails to be a CDF for $\alpha \in (0, 1)$. Instead, the model follows a construction involving *random* maxima and minima of IID random variables, preserving the spirit of the vector (S_n, M_n).

Our work begins with Section 2, which introduces new stochastic representations for random variables given by (1.4) through random maxima and minima. These leads to new representations of the variables with exponentiated distributions based on exponential, gamma, Weibull, Pareto and other models. These results are instrumental in defining a bivariate model presented in Section 3. Generalizations to random process and further extensions are presented in Section 5. The last section contains proofs and auxiliary results.

2. Univariate distributions

As before, let $\{X_i\}$ be a sequence of IID random variables with CDF F and SF S. Further, let N be a non-negative, integer-valued random variable with PDF $p(n) = \mathbb{P}(N = n)$ and probability generating function (PGF) $G_N(s) = \mathbb{E}s^N$, $|s| \leq 1$. The following lemma is crucial in obtaining representations of random variables with the SF or the CDF given in (1.4) through random maxima and minima. Here, and throughout the paper, we shall use the convention that the minimum and the maximum over an empty set are ∞ and $-\infty$, respectively.

Lemma 2.1. *In the above setting, if N is independent of the sequence $\{X_i\}$, then the CDF and the SF of the random variables*

$$X = \bigvee_{j=1}^{N} X_j \quad \text{and} \quad Y = \bigwedge_{j=1}^{N} X_j \tag{2.1}$$

are given by $F_X(x) = G_N(F(x))$ and $S_Y(y) = G_N(S(y))$, respectively.

We shall use this result to obtain representations of random variables with the SF and the CDF given in (1.4) through random maxima and minima with suitably chosen N. This cannot be accomplished by insisting that the CDF of X be $[F(x)]^\alpha$ and the SF of Y be $[S(y)]^\alpha$, since no PGF will satisfy the resulting equations,

$$G_N(F(x)) = [F(x)]^\alpha, \;\; G_N(S(y)) = [S(y)]^\alpha,$$

equivalent to $G_N(s) = s^\alpha$, unless α is an integer. To obtain a solution to this problem, one has to swap the maximum with the minimum, and instead stipulate that the SF of X be $[S(x)]^\alpha$ and the CDF of Y be $[F(y)]^\alpha$. According to Lemma 2.1, this leads to the equations

$$1 - G_N(F(x)) = [S(x)]^\alpha, \;\; 1 - G_N(S(y)) = [F(y)]^\alpha,$$

which are equivalent to

$$G_N(s) = 1 - (1-s)^\alpha, \quad |s| \le 1. \tag{2.2}$$

The above expression is indeed a genuine PGF, albeit only for $\alpha \in (0, 1]$, and represents a Sibuya random variable N_α (see, [30]) with the PDF

$$p_\alpha(n) = \mathbb{P}(N_\alpha = n) = \binom{\alpha}{n}(-1)^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(-1)^{n+1}, \quad n \in \mathbb{N}, \tag{2.3}$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$, arising in connection with discrete stable, Linnik, and Mittag-Leffler distributions (see, e.g., [4, 5, 7, 23, 25]).

Remark 2.2. Expressing the function (2.2) as a power series,

$$1 - (1-s)^\alpha = \sum_{n=1}^{\infty} \binom{\alpha}{n}(-1)^{n+1} s^n, \quad |s| \le 1, \tag{2.4}$$

makes it clear that this is not a PGF when $\alpha > 1$, as then not all coefficients of s^n in the above series are non-negative. For $\alpha \in (0, 1]$ this variable represents the number of trials till the first success in an infinite sequence of independent Bernoulli trials where the nth trial is a success with probability α/n. Clearly, at the boundary value $\alpha = 1$, the distribution is concentrated at $n = 1$. Let us also mention that in the special case $\alpha = 1/2$, we obtain the distribution of $Z+1$, where Z has a discrete Mittag-Leffler distribution with the PGF $G(s) = [1 + c(1-s)^\alpha]^{-1}$ with $c = 1$ (see, e.g., [25]).

In view of this discussion, we obtain the following result.

Corollary 2.3. *Let F be a distribution function on $\mathbb{R}$ and S be the corresponding survival function, $S(x) = 1 - F(x)$. Further, let X and Y have SF and CDF given by $[S(x)]^\alpha$ and $[F(x)]^\alpha$, respectively, where $\alpha \in (0, 1]$. Then X and Y admit the stochastic representations*

$$X = \bigvee_{j=1}^{N_\alpha} X_j \quad \text{and} \quad Y = \bigwedge_{j=1}^{N_\alpha} X_j, \tag{2.5}$$

where N_α has the Sibuya distribution (2.3) and is independent of the IID $\{X_j\}$ with the CDF F.

Let us note that that in view of (2.5) the variables X and Y from Corollary 2.3 satisfy the relation $X \ge Y$, which is consistent with the following lemma.

Lemma 2.4. *Let F be a distribution function on $\mathbb{R}$ and S be the corresponding survival function, $S(x) = 1 - F(x)$. Further, let X and Y have SF and CDF given by $[S(x)]^\alpha$ and $[F(x)]^\alpha$, respectively. Then, the variable X is stochastically larger than Y if and only if $\alpha \in (0, 1]$ and Y is stochastically larger than X if and only if $\alpha \in [1, \infty)$.*

An extension of Corollary 2.3 to the case $\alpha \in (0, \infty)$ is rather straightforward. In the following result and throughout the paper we shall use the notation

$$\{t\} = k \ \text{ and } \ \langle t \rangle = t - k \ \text{ whenever } \ k < t \leq k + 1, \ \ k = 0, 1, 2, \ldots . \tag{2.6}$$

Corollary 2.5. *Let F be a distribution function on $\mathbb{R}$ and S be the corresponding survival function, $S(x) = 1 - F(x)$. Further, let X and Y have SF and CDF given by $[S(x)]^t$ and $[F(x)]^t$, respectively, where $t \in (0, \infty)$. Then X and Y admit the stochastic representations*

$$X = \bigwedge_{j=1}^{\{t\}} X_j \wedge \bigvee_{j=\{t\}+1}^{\{t\}+N_{\langle t \rangle}} X_j \ \text{ and } \ Y = \bigvee_{j=1}^{\{t\}} X_j \vee \bigwedge_{j=\{t\}+1}^{\{t\}+N_{\langle t \rangle}} X_j, \tag{2.7}$$

where $N_{\langle t \rangle}$ has the Sibuya distribution (2.3) with parameter $\alpha = \langle t \rangle$ and is independent of the IID $\{X_j\}$ with the CDF F.

3. Bivariate models

An extension to bivariate distribution having margins with SF and CDF given by (1.4) is straightforward. One such distribution is obtained by putting together the X and Y defined in Corollary 2.3. Let us start with the following result, which extends Lemma 2.1 to the bivariate setting.

Lemma 3.1. *Let*

$$(X, Y) = \left(\bigvee_{j=1}^{N} X_j, \bigwedge_{j=1}^{N} X_j \right), \tag{3.1}$$

where N is an integer-valued random variable N supported on $\mathbb{N} = \{1, 2, \ldots\}$ and the $\{X_i\}$ are IID random variables, independent of N. Then, the CDF of (X, Y) is given by

$$F(x, y) = \begin{cases} G_N(F(x)) - G_N(F(x) - F(y)) & \text{for } x > y \\ G_N(F(x)) & \text{for } x \leq y, \end{cases} \tag{3.2}$$

where $G_N(\cdot)$ and $F(\cdot)$ are the PGF of N and the common CDF of the $\{X_i\}$, respectively.

According to Lemma 2.1, the marginal CDF of X in (3.1) is $F_X(x) = G_N(F(x))$ while the marginal SF of Y in (3.1) $S_Y(y) = G_N(S(y))$, where $S(y) = 1 - F(y)$ is the common SF of the $\{X_i\}$. Taking N to be Sibuya distributed N_α given by the PGF (2.2) we obtain a bivariate distribution with marginal SF of X and the marginal CDF of Y given by $[S(x)]^\alpha$ and $[F(x)]^\alpha$, respectively. The following result summarizes this.

Corollary 3.2. *Let*

$$(X,Y) = \left(\bigvee_{j=1}^{N_\alpha} X_j, \bigwedge_{j=1}^{N_\alpha} X_j \right), \tag{3.3}$$

where N_α is Sibuya distributed given by the PGF (2.2) with $\alpha \in (0,1]$ and the $\{X_i\}$ are IID random variables, independent of N_α. Then, the joint CDF of (X,Y) is given by

$$F(x,y) = \begin{cases} [1-F(x)+F(y)]^\alpha - [1-F(x)]^\alpha & \text{for } x > y \\ 1-[1-F(x)]^\alpha & \text{for } x \le y, \end{cases} \tag{3.4}$$

the marginal SF of X is $S_X(x) = [S(x)]^\alpha$, while the marginal CDF of Y is $F_Y(y) = [F(x)]^\alpha$, where $S(\cdot)$ and $F(\cdot)$ are the common SF and CDF of the $\{X_i\}$, respectively.

Remark 3.3. In the special case $\alpha = 1$, the joint CDF (3.4) becomes

$$F(x,y) = \begin{cases} F(y) & \text{for } x > y \\ F(x) & \text{for } x \le y, \end{cases} \tag{3.5}$$

and it describes the random vector (X,X), to which (3.3) reduces in this case (since $N_\alpha = 1$ when $\alpha = 1$). Note that this CDF is also a special case $n = 1$ of the joint CDF of the two extreme order statistics, S_n and M_n, given by (1.3).

Observe that replacing the n in the joint CDF (1.3) of the two extreme order statistics with $\alpha \in (0,1)$ does not produce the CDF (3.4). Moreover, the resulting quantity is not even a proper CDF, as can be easily verified. Our next result provides a generalization of the bivariate CDF (1.3) to non-integer values of n.

Proposition 3.4. *Let $t = n + \alpha \in (0, \infty)$, where $n = \{t\} \in \{0\} \cup \mathbb{N}$ and $\alpha = \langle t \rangle \in (0, 1]$, and define*

$$(X, Y) = \left(\bigwedge_{j=1}^{n} X_j \wedge \bigvee_{j=n+1}^{n+N_\alpha} X_j, \bigvee_{j=1}^{n} X_j \vee \bigwedge_{j=n+1}^{n+N_\alpha} X_j \right), \tag{3.6}$$

where N_α has the Sibuya distribution (2.3), and is independent of the IID $\{X_j\}$ with the CDF F. Then, the joint CDF of (X, Y) is

(3.7)

$$F(x, y) = \begin{cases} [F(y)]^{n+\alpha} - \{F(y) - F(x)\}^n \{[F(y)]^\alpha - 1 + [1 - F(x)]^\alpha\} & \text{for } x < y \\ [F(y)]^{n+\alpha} - \{[F(y)]^\alpha - [1 - F(x) + F(y)]^\alpha + [1 - F(x)]^\alpha\} \mathbb{I}_{\{0\}}(n) & \text{for } x \geq y, \end{cases}$$

where $\mathbb{I}_A(\cdot)$ is the indicator function of the set A.

Remark 3.5. Note that by Corollary 2.5, the marginal SF of X is $[S(x)]^t$ and the marginal CDF of Y is $[F(y)]^t$, where $S(\cdot)$ and $F(\cdot)$ are the common SF and CDF of the $\{X_i\}$, respectively. This can also be seen by taking the limits $x \to \infty$ or $y \to \infty$ in (3.7). Moreover, when $t \in \mathbb{N}$, the CDF (3.7) coincides with (1.3). Additionally, when $n = 0$, the indicator function in (3.7) takes on the value of 1, and we obtain (3.4). Formally, the CDF (3.7) is well defined for any non-negative integer n and any $\alpha \in [0, 1]$. For the boundary cases $\alpha = 0$ or $\alpha = 1$, we recover the bivariate CDF of the extreme order statistics.

If the bivariate distributions discussed above are continuous, they can also be described through the copula representation (see [31])

$$F(x, y) = C(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}.$$

Straightforward calculations show that the copula C_N connected with the bivariate distribution (3.2) is given by

$$C_N(u, v) = \begin{cases} u - G_N(G_N^{-1}(u) + G_N^{-1}(1 - v) - 1) & \text{for } G_N^{-1}(u) + G_N^{-1}(1 - v) > 1 \\ u & \text{for } G_N^{-1}(u) + G_N^{-1}(1 - v) \leq 1. \end{cases} \tag{3.8}$$

In the Sibuya case, the copula $C_\alpha = C_{N_\alpha}$ connected with the bivariate distribution (3.3) is given by

$$C_\alpha(u, v) = \begin{cases} u - 1 + \{(1 - u)^{1/\alpha} + v^{1/\alpha}\}^\alpha & \text{for } (1 - u)^{1/\alpha} + v^{1/\alpha} < 1 \\ u & \text{for } (1 - u)^{1/\alpha} + v^{1/\alpha} \geq 1. \end{cases} \tag{3.9}$$

Similarly, in the more general case of (3.6) with $n \geq 1$, the copula is

(3.10)
$$C_{\alpha,n}(u,v) = \begin{cases} v & \text{for } (1-u)^{\frac{1}{\alpha+n}} + v^{\frac{1}{\alpha+n}} \leq 1 \\ v - \{(1-u)^{\frac{1}{\alpha+n}} + v^{\frac{1}{\alpha+n}} - 1\}^n \{(1-u)^{\frac{\alpha}{\alpha+n}} + v^{\frac{\alpha}{\alpha+n}} - 1\} & \text{otherwise.} \end{cases}$$

This is a generalization of the min-max copula discussed in [28], to which it reduces for boundary values $\alpha = 0, 1$.

4. Further examples and a duality of distributions on $\mathbb{N}$

Numerous new families of distributions can be generated from a "base" CDF $F(\cdot)$ (or SF $S(\cdot)$) via Lemma 2.1 with suitably chosen distribution of N. One well-known example of such a construction is the so-called Marshall-Olkin scheme introduced in [20], where a SF S generates a family of SFs given by

$$S(x;\alpha) = \frac{\alpha S(x)}{1-(1-\alpha)S(x)}, \;\; x \in \mathbb{R}, \;\; \alpha \in \mathbb{R}_+. \tag{4.1}$$

For $\alpha = p \in (0,1)$, the above SF is of the form $G_N(S(x))$, where G_N is the PGF of a geometric random variable N with parameter p, given by the PDF

$$\mathbb{P}(N = n) = p(1-p)^{n-1}, \;\; n \in \mathbb{N}. \tag{4.2}$$

A straightforward calculation shows that, when $\alpha > 1$, then the CDF corresponding to (4.1) is of the form $G_N(F(x))$, where $F(x) = 1 - S(x)$ and G_N is the PGF of a geometric random variable with parameter $p = 1/\alpha$. Thus, in view of Lemma 2.1, the family of distributions defined viz. (4.1) is generated by geometric minima ($0 < \alpha < 1$) and geometric maxima ($\alpha > 1$) of IID random variables with the base SF S, as noted in [20].

Another example of this construction is connected with the so-called *transmuted distributions*, which have populated the literature since the introduction of the *quadratic rank transmutation map* in [29]. The latter, defined via

$$u \to u + \alpha u(1-u), \;\; u \in [0,1], \alpha \in [-1,1], \tag{4.3}$$

is used to transform a base CDF F into its transmuted version F_α, where

$$F_\alpha(x) = (1+\alpha)F(x) - \alpha F^2(x), \;\; x \in \mathbb{R}. \tag{4.4}$$

As recently shown in [17], the CDF in (4.4) is of the form $G_{-\alpha}(F(x))$ for $\alpha \in [-1, 0]$ and of the form $1 - G_\alpha(1 - F(x))$ for $\alpha \in [0, 1]$, where

$$G_p(s) = s(1 - p + ps), \quad s \in [0, 1], \tag{4.5}$$

is the PGF of a Bernoulli distribution with parameter $p \in [0, 1]$ shifted up by one. Again, by Lemma 2.1, the family of distributions defined viz. (4.4) is generated by random minima ($0 \leq \alpha \leq 1$) and random maxima ($-1 \leq \alpha \leq 0$) of IID random variables with the base CDF F. See [17] for more information and extensive references.

The above examples illustrate the fact that virtually any integer-valued random variable N generates a new probability distribution from that of a base random variable X connected with the maximum (or minimum) of N IID copies of X. In view of this, we pose the following question: *If X is coupled with N as in Lemma 2.1 to generate a new distribution of $Y \stackrel{d}{=} \max\{X_1, \ldots, X_N\}$, then can this process be reversed, so that for some integer-valued random variable $\tilde{N}$ we would also have $X \stackrel{d}{=} \min\{Y_1, \ldots, Y_{\tilde{N}}\}$, where the $\{Y_i\}$ are IID copies of Y, independent of $\tilde{N}$?* Similarly, if the new distribution is generated via a random minimum rather than maximum, so that, $Y \stackrel{d}{=} \min\{X_1, \ldots, X_N\}$, then is there an integer-valued random variable $\tilde{N}$ so that we would have $X \stackrel{d}{=} \max\{Y_1, \ldots, Y_{\tilde{N}}\}$? In view of of our results in Section 2, it is not hard to see that in order for these conditions to hold the PGFs of N and $\tilde{N}$ need to satisfy the relation

$$G_{\tilde{N}}(1 - G_N(s)) = 1 - s, \; s \in [0, 1], \tag{4.6}$$

so that

$$G_{\tilde{N}}(s) = 1 - G_N^{-1}(1 - s), \; s \in [0, 1]. \tag{4.7}$$

In turn, if the quantity on the right-hand-side of (4.7) is a genuine PGF, then the process of taking random maxima (or minima) can be "reversed" as described above. This motivates the following definition.

Definition 4.1. Let N be a random variable supported on $\mathbb{N}$ with the PGF G_N. If the quantity in (4.7) is a genuine PGF, then the corresponding random variable $\tilde{N}$ supported on $\mathbb{N}$ is said to be a *dual* to N with respect to the operation of random min/max.

Note that the notion of being a dual is symmetric, that is if $\tilde{N}$ is a dual to N then in turn N is a dual of $\tilde{N}$. Further, as noted above, the four operations of taking random min/max with either N or $\tilde{N}$ are reversible and recover the original distribution, as stated in the following result.

Proposition 4.2. *If N and $\tilde{N}$ are a dual pair of random variables supported on $\mathbb{N}$, then for any double-sequence $\{X_{ij}\}$ of IID random variables we have*

$$X_{11} \stackrel{d}{=} \bigwedge_{i=1}^{N} \bigvee_{j=1}^{\tilde{N}_i} X_{ij} \stackrel{d}{=} \bigvee_{i=1}^{N} \bigwedge_{j=1}^{\tilde{N}_i} X_{ij} \stackrel{d}{=} \bigwedge_{i=1}^{\tilde{N}} \bigvee_{j=1}^{N_i} X_{ij} \stackrel{d}{=} \bigvee_{i=1}^{\tilde{N}} \bigwedge_{j=1}^{N_i} X_{ij}, \tag{4.8}$$

with the $\{X_{ij}\}$ being independent of all the integer valued random variables that appear in (4.8).

Let us now present several dual pairs related to Sibuya and other standard probability distributions.

4.1. **Deterministic/Sibuya pair.** If the random variable N is a deterministic one, taking on the value of $n \in \mathbb{N}$ with probability 1, then the PGF of N is $G_N(s) = s^n$ while the dual PGF in (4.7) becomes

$$G_{\tilde{N}}(s) = 1 - (1-s)^{1/n}, \; s \in [0,1], \tag{4.9}$$

which we recognize to be Sibuya distributed with parameter $1/n \in (0,1]$. In turn, the dual to a Sibuya variable with parameter $1/n$ is a deterministic variable equal to n. However, if N is taken to be Sibuya with parameter α where $1/\alpha$ is not an integer, then the quantity $G_{\tilde{N}}(s) = s^{1/\alpha}$ is not a valid PGF. Thus, a Sibua random variable has no dual unless $1/\alpha$ is an integer.

4.2. **Shifted Bernoulli/tilted Sibuya pair.** Let N have a Bernoulli distribution with parameter $p \in [0,1]$ shifted up by one, given by the PGF (4.5). Then routine albeit lengthy calculations lead to the dual PGF,

$$G_{\tilde{N}}(s) = \frac{1+p}{2p}\left\{1 - \left(1 - \frac{4p}{(1+p)^2}s\right)^{\frac{1}{2}}\right\}, \; s \in [0,1], \tag{4.10}$$

which, upon expending the square root above into a power series, produces the dual PDF

$$\mathbb{P}(\tilde{N}=n)=\frac{1+p}{2p}\left[\frac{4p}{(1+p)^2}\right]^n\binom{1/2}{n}(-1)^{n+1},\ \ n\in\mathbb{N}. \tag{4.11}$$

This can be seen as an exponentially tilted Sibuya distribution of N_α given by (2.3) with $\alpha=1/2$,

$$\mathbb{P}(\tilde{N}=n)=\frac{r^n\mathbb{P}(N_\alpha=n)}{\mathbb{E}r^{N_\alpha}},\ \ n\in\mathbb{N}, \tag{4.12}$$

where $r=4p/(1+p)^2$.

4.3. **Poisson/logarithmic pair.** Consider a Poisson random variable with parameter λ truncated below at 1, given by the PDF

$$\mathbb{P}(N=n)=\frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})},\ \ n\in\mathbb{N}, \tag{4.13}$$

and the PGF

$$G_N(s)=\frac{e^{-\lambda(1-s)}-e^{-\lambda}}{1-e^{-\lambda}},\ \ s\in[0,1]. \tag{4.14}$$

Straightforward algebra shows that the dual PGF in (4.7) becomes

$$G_{\tilde{N}}(s)=-\frac{1}{\lambda}\log\left\{1-(1-e^{-\lambda})s\right\},\ \ s\in[0,1], \tag{4.15}$$

which, upon expending the logarithm above into a power series, is seen to describe a *logarithmic* distribution with the PDF

$$\mathbb{P}(\tilde{N}=n)=-\frac{(1-q)^n}{n\log q},\ \ n\in\mathbb{N},\ \ q=e^{-\lambda}\in(0,1). \tag{4.16}$$

4.4. **Negative binomial/negative binomial pair.** Consider a negative binomial (NB) distribution with parameters $r\in\mathbb{R}_+$ and $p\in(0,1]$, given by the PDF

$$p_n=\binom{r+n-1}{r}p^r(1-p)^n,\ \ n\in\mathbb{N}_0=\{0,1,\ldots\}. \tag{4.17}$$

Let N have the above distribution truncated below at 1, with the probabilities given by $p_n/(1-p^r)$, $n\in\mathbb{N}$, and the PGF

$$G_N(s)=\frac{1}{1-p^r}\left\{\left[\frac{p}{1-(1-p)s}\right]^r-p^r\right\},\ \ s\in[0,1]. \tag{4.18}$$

After routine calculation, we find the dual PGF to be

$$G_{\tilde{N}}(s) = \frac{1}{1-p}\left\{\left[\frac{p^r}{1-(1-p^r)s}\right]^{\frac{1}{r}} - p\right\}, \quad s \in [0,1], \tag{4.19}$$

which we recognize to be a NB distribution with parameters $1/r$ and p^r, truncated below at 1. Thus, NB distributions with parameters r, p and $1/r, p^r$, both truncated below at 1, form a dual pair in our scheme. In particular, when $r = 1$, both distributions of the dual pair are the same geometric distributions, given by the PDF (4.2). This has an interesting interpretation: If N is geometrically distributed and Y is obtained via a maximum (minimum) of N IID variables $\{X_i\}$ and in turn we take the minimum (maximum) of N IID copies of Y, then we recover the distribution of the $\{X_i\}$.

5. Extensions to random processes

The Sibuya distribution with parameter $t = \alpha$, $t \in [0,1]$, can be obtained as the marginal distribution of a Sibuya random process on $[0,1]$,

$$N(t) = \min\{n \in \mathbb{N} : U_n \leq t/n\}, \ t \in [0,1], \tag{5.1}$$

which was introduced in [18]. Here, the $\{U_n\}$ are IID standard uniform random variables and $N(0) = \infty$. Clearly, since $\mathbb{P}(nU_n \leq t) = t/n$ for each $t \in (0,1]$, the variable $N(t)$ has the Sibuya distribution (2.3) with $\alpha = t$. Moreover, as shown in [18], this is a right-continuous, pure-jump, and non-increasing process, with an infinite number of jumps in any (right) neighborhood of zero. In addition, the location of the jumps and their sizes are closely related to the magnitudes and the locations of the *records* connected with the sequence $\{nU_n\}$, where the value that is *smaller* than all the previous values sets a new record. These are described via the pairs K_i and R_i, which are the time and the size, respectively, of the ith record among the $\{nU_n\}$. By assumption, the first value is a record, so that $K_1 = 1$ and $R_1 = U_1$. Since U_1 is less than one with probability one, all the $\{R_i\}$ are less than one as well. Further, we let $\delta_i = R_{i-1} - R_i$ (with $R_0 = 1$) represent the differences between successive record values while $\tau_i = K_i - K_{i-1}$ are the inter-arrival times between successive records. With this notation, the Sibuya process (5.1) admits the representation

$$N(t) = 1 + \sum_{i=1}^{\infty} \tau_{i+1} \mathbb{I}_{(t,1]}(r_i), \;\; t \in [0,1], \tag{5.2}$$

where $\mathbb{I}_A(\cdot)$ is the indicator function of the set A. Viewing from right to left, the random process $N(t)$ starts with $N(1) = 1$ and then jumps up by τ_{i+1} at every record value R_i. Other than that, the values of $N(t)$ are constant on the intervals $[R_n, R_{n-1})$, and $N(R_n) = K_n$. Twenty samples of the Sibuya process is presented in Figure 1. More information about this process can be found in [18].

The above Sibuya process can be used to construct a bivariate extremal process on $[0,1]$ via Corollary 3.2. Starting with a sequence $\{X_n\}$ of IID random variables with the common CDF F and SF S, we let

$$(X(t), Y(t)) = \left(\bigvee_{j=1}^{N(t)} X_j, \bigwedge_{j=1}^{N(t)} X_j \right), \;\; t \in [0,1], \tag{5.3}$$

where $N(t)$ is the Sibuya process, independent of the $\{X_n\}$. It follows that for each $t \in (0,1)$ the marginal distributions of $(X(t), Y(t))$ are given by (3.4) with $\alpha = t$. In particular, the marginal SF of $X(t)$ is $[S(\cdot)]^t$, while the marginal CDF of $Y(t)$ is $[F(\cdot)]^t$. Below we extend this construction beyond the unit interval using Proposition 3.4.

5.1. **A random record process.** Consider a sequence $\mathbf{X} = \{X_n\}$ of IID, non-negative random variables, and the corresponding upper and the lower records based on these variables, denoted by $(K_i^U(\mathbf{X}), R_i^U(\mathbf{X}))$ and $(K_i^L(\mathbf{X}), R_i^L(\mathbf{X}))$, respectively. Thus, K_i^U is the time (index) at which the ith record occurs among the $\{X_i\}$, and $R_i^U = X_{K_i^U}$ is the size of that record, where a value that is *larger* than all the previous values sets a new record. The quantities K_i^L and $R_i^L = X_{K_i^L}$ are defined similarly, when a new record is the value that is *smaller* than the ones previously observed.

We propose a continuous time process $\{(X(t), Y(t)), \; t > 0\}$, containing full information about the records connected with the $\{X_n\}$, and having certain invariance properties that in our opinion justify naming it *the record process*. Embedding the records in the record process is similar to embedding the minima S_n or the maxima M_n (given by (1.1) and (1.2), respectively), into the extremal processes, as described in [27]. However, despite the fact the record process and the extremal processes have the same marginal distributions, they are essentially different, as it will be shown below.

The record process connected with the sequence $\mathbf{X} = \{X_n\}$ is defined through a sequence $\{N^{(i)}(t)\}$, $i \in \mathbb{N}$, of independent Sibuya processes, which are independent of $\mathbf{X}$. Here, each $\{N^{(i)}(t)\}$ is connected with an underlying sequences $U_{i,n}$ ($n \in \mathbb{N}$) of IID standard uniform random variables, as discussed above. As in Section 2, a real t is split into its integer part $\{t\}$ and its fractional part $\langle t \rangle = t - \{t\}$, defined in (2.6). Subsequently, we set

$$(5.4)\quad (X(t), Y(t)) = \left(\bigwedge_{j=1}^{\{t\}} X_j \wedge \bigvee_{i=1}^{N^{(\{t\})}(\langle t \rangle)} X_{\{t\}+i}, \bigvee_{j=1}^{\{t\}} X_j \vee \bigwedge_{i=1}^{N^{(\{t\})}(\langle t \rangle)} X_{\{t\}+i} \right), \quad t \in (0, \infty).$$

For $t \in (0,1)$, where we have $\{t\} = 0$ and $\langle t \rangle = t$, the minimum $\bigwedge_{j=1}^{0} X_j$ and the maximum $\bigvee_{j=1}^{\{t\}} X_j$ above are understood as ∞ and $-\infty$, respectively, and $(X(t), Y(t))$ has the same structure as (5.3). The component $X(t)$ of this construction connects with the lower records $(K_i^L(\mathbf{X}), R_i^L(\mathbf{X}))$ of the sequence $\mathbf{X} = \{X_n\}$, while the component $Y(t)$ relates to its upper records $(K_i^U(\mathbf{X}), R_i^U(\mathbf{X}))$. Moreover, by Proposition 3.4, the bivariate marginal distributions of $(X(t), Y(t))$ are given by (3.7) with $n = \{t\}$ and $\alpha = \langle t \rangle$. In particular, the the marginal SF of $X(t)$ is $[S(\cdot)]^t$, while the marginal CDF of $Y(t)$ is $[F(\cdot)]^t$ for each $t \in \mathbb{R}_+$. The following result summarizes fundamental properties of the record process (5.4).

Theorem 5.1. *Let $(X(t), Y(t))$ be a random record process (5.4) connected with a sequence $\mathbf{X} = \{X_n\}$ of IID, non-negative random variables with the CDF F and the SF S. Then we have the following:*

(i) *For $t \in \mathbb{R}_+$, the SF of $X(t)$ is given by $S^t(\cdot)$ while the CDF of $Y(t)$ is given by $F^t(\cdot)$.*

(ii) *For $t \in \mathbb{R}_+$, the joint distributions of $(X(t), Y(t))$ is given by (3.7) with $n = \{t\}$ and $\alpha = \langle t \rangle$.*

(iii) *Both, $X(t)$ and $Y(t)$ are a pure jump processes, with right-hand-side continuous trajectories. Moreover, the trajectories of $X(t)$ are non-increasing, while the trajectories of $Y(t)$ are non-decreasing.*

(iv) *If $t = n \in \mathbb{N}$, then $X(n) = \bigwedge_{k=1}^{n} X_k$ and $Y(n) = \bigvee_{k=1}^{n} X_k$ with probability one.*

(v) *If* $t_n \in (n, n+1)$ *is the location of the last jump of* X *on* $(n, n+1)$*, then* X *on* (n, t_n) *is independent of* $X(t)$ *on* $\mathbb{R}^+ \setminus (n, t_n)$*. Similar property holds for the* $Y(t)$*.*

Remark 5.2. Let us note that, in contrast with the extremal processes described in [27], here $X(t)$ and $Y(t)$ are directly related to the maxima and the minima of the underlying sequence $\{X_n\}$. Moreover, if the value of X_n $(n > 1)$ does not set a new record, so that $S_{n-1} \leq X_n \leq M_{n-1}$, the trajectories of $X(t)$ and $Y(t)$ are constant on the interval $[n-1, n]$. If X_n does set a new record with $X_n > M_{n-1}$, then the trajectory of $X(t)$ is still constant (and equal to S_{n-1}) on $[n-1, n]$ while that of $Y(t)$ will be increasing from $Y(n-1) = M_{n-1}$ to $Y(n) = M_n$ through a sequence of jumps on the interval $[n-1, n]$, with both, the jump sizes and their locations, affected by the "future" values $\{X_i\}_{i=n}^{\infty}$ of the underlying sequence $\mathbf{X}$. Similarly, If X_n sets a new downward record with $X_n < S_{n-1}$, then the trajectory of $Y(t)$ will be constant (and equal to M_{n-1}) on the interval $[n-1, n]$ while that of $X(t)$ will be decreasing from $X(n-1) = S_{n-1}$ to $X(n) = S_n$ through a sequence of jumps on $[n-1, n]$, related to the future values of $\mathbf{X}$. Interestingly, if a trajectory of either component of our process does go up (or down) in $(n-1, n)$, one can predict a record value is to occur at time $t = n$.

While a complete analysis of the bivariate process (5.4) is beyond the scope of this paper, below we provide few additional remarks shedding more light on its basic properties. First, let us contrast the properties of the introduced process with some misleading intuition following from extreme value theory. Let us start with a discussion of the structure of the Sibuya process $N(t)$, discussed fully in [18]. Namely, the arrival times $(R_1, \ldots, R_N)$ of the first N jumps can be represented as $\left(\prod_{i=1}^{N} U_i, \cdots, U_1\right)$, where the $\{U_i\}$ are IID standard uniform variables. The jump sizes (viewed from left to right) can be iteratively computed from the following scheme. We take $\tau_N = 1$ as the first value of the jump at $R_N = r_N$, and then recursively evaluate the jumps at the locations r_{N-k}, $k \geq 1$, by taking $\tau_{N-k} = [\log(V_k)/\log(1-p_k)] + 1$, where the $\{V_k\}$ are IID uniformly distributed (independently of everything else), while the $\{p_k\}$ are sampled independently from a beta distribution with parameters $(1-r_{N-k+1}, r_{N-k+1}+\tau_{N-k+1})$. Due to extremely large values that the process $N(t)$ takes in the vicinity of zero, this scheme is much more effective in

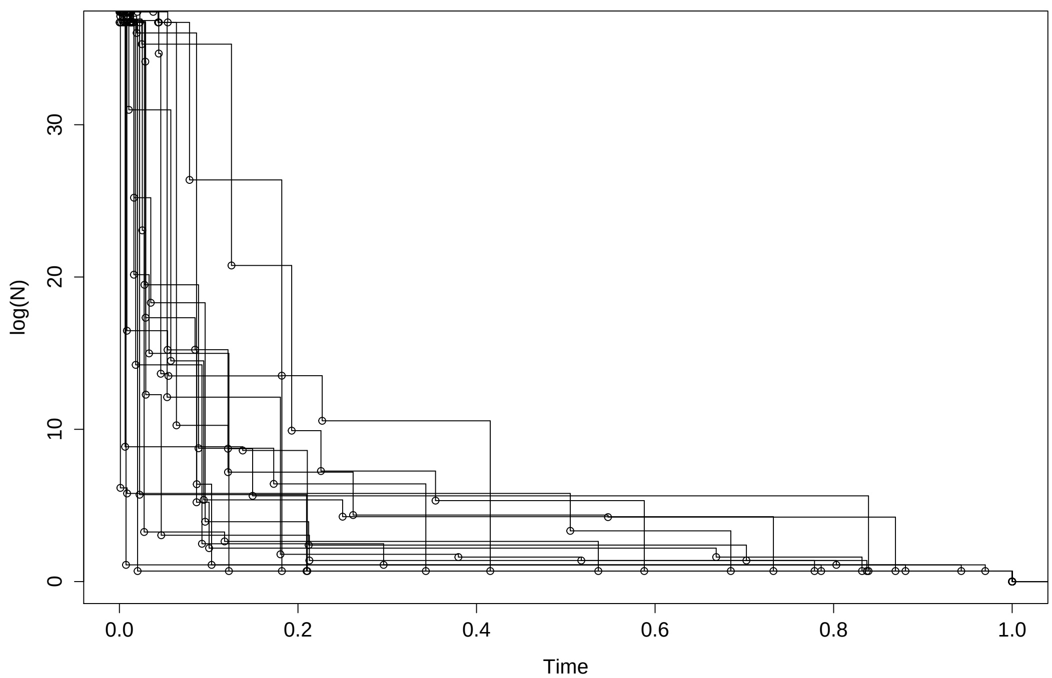

FIGURE 1. Twenty samples from the Sibuya process on the logarithmic scale in values. The abscissae of the circles indicate locations of jumps and their ordinates the logarithmic values of the Sibuya process after the jump

simulating values of the Sibuya process than directly from (5.1), which would require an enormous number of uniform variables to be simulated. Nevertheless, typically there are only very few jumps that can be realistically observed in the process due to their extremely large values when process nears in the argument to zero. This can be seen from Figure 1, where twenty samples from the process are given on the logarithmic scale in values.

As we stated above, the jumps of the process $N(t)$ are extremely large. In our simulations, they were on the order of 10^{10} or higher for $t < 0.05$. Thus, based on classical extreme value theory, one might expect that the extremal process given in (5.3) would exhibit some asymptotical distributional invariance. Namely, by extremal types theorem, under certain conditions on the $\{X_j\}$ and when appropriately normalized, the maximum $M_n = \bigvee_{j=1}^{n} X_j$

converges in distribution to one of the three standard extreme value distributions. For example, if the $\{X_j\}$ were standard Pareto distributed with the SF $S(x) = 1 \wedge x^{-\alpha}$, then the limiting distribution of $M_n/n^{1/\alpha}$ is the Fréchet distribution given by the CDF $\exp(-x^{-\alpha})$, $x > 0$. However despite the fact that $N(t)$ is very large when t is close to zero, the process $X(t) = \bigvee_{j=1}^{N(t)} X_j$ does not lose the effect of the entire SF of X on its SF, which is $[S(x)]^t$. In our special case, this is still Pareto distribution, but with the tail parameter αt, and thus with much heavier power tail than that of the Fréchet distribution.

The above considerations lead to a question about the asymptotics of $X(t)$ at zero. Namely, are there any normalizing functions $a(t)$ and $b(t)$ such

$$Z(t) = (X(t) - b(t))/a(t) \tag{5.5}$$

has a non-degenerate limiting distribution as t approaches zero? This is normally the case when the random index $N(t)$ tends to infinity while a scaled version of it, $c(t)N(t)$, has a non-zero limiting distribution (see, e.g., [8], Section 4.3). However, as shown below, the Sibuya distribution $N(t)$ increases so fast as t gets close to zero, that no normalization $c(t)$ can make it converge to a non-zero limit.

Proposition 5.3. *For $p \in (0,1)$, let N_p have a Sibua distribution (2.3) with parameter $\alpha = p$, and suppose that $c_p N_p \stackrel{d}{\to} Z$ as $p \to 0$ for some $c_p > 0$. Then $\mathbb{P}(Z = 0) = 1$.*

In view of the above result, one may suspect that the quantity $Z(t)$ in (5.5) can never converge to an extreme value (or other non-degenerate) distribution as t approaches zero. As shown below, this is indeed the case. To see this, consider again standard Pareto distributed $\{X_i\}$ and assume to the contrary that there exist functions $a(t)$ and $b(t)$ such that the distribution of $Z(t)$ has a non-degenerate limit at zero. If this was so, then $a(t)$ would have to converge to infinity, as otherwise $b(t)$ would have to converge to infinity to ensure that the SF $1 \wedge (xa(t)+b(t))^{-\alpha t}$ would converge to a SF. However such a limit would be independent of x, and thus it would not constitute a proper distribution. Further, note that $b(t)/a(t)$ must converge to infinity as well, since otherwise $1 \wedge (xa(t) + b(t))^{-\alpha t} = 1 \wedge x^{-\alpha t} a(t)^{-\alpha t} \left(1 + \frac{b(t)}{xa(t)}\right)^{-\alpha t}$ would converge to a constant independent of x. However, if both $a(t)$ and $b(t)/a(t)$ converge to infinity, then $1 \wedge x^{-\alpha t} b(t)^{-\alpha t} \left(\frac{1}{x} + \frac{a(t)}{b(t)}\right)^{-\alpha t}$ will have a limit not dependent on x and thus not a proper distribution function. We conclude that no the extremal types theorem holds for $X(t) = \bigvee_{j=1}^{N(t)} X_j$ as t converges to zero.

6. Proofs

Proof of Lemma 2.1. This result follows from a standard conditioning argument. Indeed, since

$$\mathbb{P}(\bigvee_{j=1}^{N} X_j \le x) = \sum_n \mathbb{P}(\bigvee_{j=1}^{N} X_j \le x|N = n)p_n = \sum_n \mathbb{P}(\bigvee_{j=1}^{n} X_j \le x)p_n = \sum_n [F(x)]^n p_n,$$

we obtain $F_X(x) = G_N(F(x))$. Similarly,

$$\mathbb{P}(\bigwedge_{j=1}^{N} X_j > y) = \sum_n \mathbb{P}(\bigwedge_{j=1}^{N} X_j > y|N = n)p_n = \sum_n \mathbb{P}(\bigwedge_{j=1}^{n} X_j > y)p_n = \sum_n [S(y)]^n p_n,$$

leads to $S_Y(y) = G_N(S(y))$, and completes the proof. □

Proof of Lemma 2.4. Recall that X is stochastically larger than Y if and only if the CDF of X is always less or equal than the CDF Y, which in our case is equivalent to $1-[S(x)]^\alpha \le [F(x)]^\alpha$ for all $x \in \mathbb{R}$. Let $g_\alpha(u) = u^\alpha + (1-u)^\alpha, \; u \in [0,1]$. Then, the above inequality is equivalent to $g_\alpha(u) \ge 1$ for all $u \in [0,1]$. It is easy to see that the latter holds if and only if $\alpha \in (0,1]$. Indeed, the inequality holds trivially when $\alpha = 1$. Moreover, we have $g_\alpha(0) = g_\alpha(1) = 1$ and the function g_α is increasing on $(0,1/2)$ and decreasing on $(1/2,1)$ whenever $\alpha \in (0,1)$, in which case $g_\alpha(u) \ge g(1) = 1$ for all $u \in [0,1]$. This proofs the first part of the lemma. Similarly, Y is stochastically larger than X if and only if $1-[S(x)]^\alpha \ge [F(x)]^\alpha$ for all $x \in \mathbb{R}$, which in terms of g_α means that $g_\alpha(u) \le 1$ for all $u \in [0,1]$. This again holds trivially when $\alpha = 1$, while for $\alpha \in (1,\infty)$ the function g_α is decreasing on $(0,1/2)$ and increasing on $(1/2,1)$, leading to $g_\alpha(u) \le g(1) = 1$ in this case. This concludes the proof. □

Proof of Lemma 3.2. Write $F(x,y) = \mathbb{P}(X \le x) - \mathbb{P}(X \le x, Y > y)$, which, by Lemma 2.1, leads to $F(x,y) = G_N(F(x)) - \mathbb{P}(X \le x, Y > y)$. Next, we consider the term $\mathbb{P}(X \le x, Y > y)$. Clearly, this reduces to zero whenever $x \le y$ (since, by the definition of these variables, we have $X \geq Y$). In turn, for $x > y$, by conditioning on N we obtain

$$\mathbb{P}(X \leq x, Y > y) = \sum_{n=1}^{\infty} \mathbb{P}(\bigvee_{j=1}^{N} X_j \leq x, \bigwedge_{j=1}^{N} X_j > y | N = n)\mathbb{P}(N = n).$$

Since the $\{X_i\}$ and N are independent, we have

$$P(\bigvee_{j=1}^{N} X_j \leq x, \bigwedge_{j=1}^{N} X_j > y | N = n) = P(\bigvee_{j=1}^{n} X_j \leq x, \bigwedge_{j=1}^{n} X_j > y) = [\mathbb{P}(y < X_j \leq x)]^n,$$

so that

$$\mathbb{P}(X \leq x, Y > y) = \sum_{n=1}^{\infty} [F(x) - F(y)]^n \mathbb{P}(N = n) = G_N(F(x) - F(y)).$$

This completes the proof. ☐

Proof of Proposition 3.4. Write

$$(\tilde{X}, \tilde{Y}) = \left(\bigwedge_{j=1}^{n} X_j, \bigvee_{j=1}^{n} X_j \right) \text{ and } (X_\alpha, Y_\alpha) = \left(\bigvee_{j=n+1}^{n+N_\alpha} X_j, \bigwedge_{j=n+1}^{n+N_\alpha} X_j \right),$$

noting that $(X, Y) = (\tilde{X} \wedge X_\alpha, \tilde{Y} \vee Y_\alpha)$ and the two random vectors above are independent. Using this representation allows for expressing the joint CDF of (X, Y) as

$$F(x, y) = \mathbb{P}(\tilde{Y} \vee Y_\alpha \leq y) - \mathbb{P}(\tilde{X} \wedge X_\alpha > x, \tilde{Y} \vee Y_\alpha \leq y). \tag{6.1}$$

By independence of $\tilde{Y}$ and Y_α, the first term on the right-hand-side above can be written as

$$\mathbb{P}(\tilde{Y} \vee Y_\alpha \leq y) = \mathbb{P}(\tilde{Y} \leq y, Y_\alpha \leq y) = [F(y)]^{n+\alpha},$$

since the marginal CDFs of $\tilde{Y}$ and Y_α are given by $[F(y)]^n$ and $[F(y)]^\alpha$, respectively. Next, we write the second term on the right-hand-side in (6.3) as

$$\mathbb{P}(\tilde{X} > x, X_\alpha > x, \tilde{Y} \leq y, Y_\alpha \leq y) = \mathbb{P}(\tilde{X} > x, \tilde{Y} \leq y)\mathbb{P}(X_\alpha > x, Y_\alpha \leq y), \tag{6.2}$$

which is equivalent to

$$[\mathbb{P}(\tilde{Y} \leq y) - \mathbb{P}(\tilde{X} \leq x, \tilde{Y} \leq y)][\mathbb{P}(Y_\alpha \leq y) - \mathbb{P}(X_\alpha \leq x, Y_\alpha \leq y)].$$

To complete the proof, substitute the CDFs of $(\tilde{X}, \tilde{Y})$ and (X_α, Y_α), which are given by (1.3) and (3.4), respectively. ☐

Proof of Theorem 5.1. Let $Y_1(t) = \bigvee_{k=1}^{\{t\}} X_k$ and $Y_2(t) = \bigwedge_{i=1}^{N^{\{t\}}(\langle t\rangle)} X_{\{t\}+i}$. It is clear that $Y_1(t)$ and $Y_2(t)$ are independent for each fixed t so that by Corollary 2.3

$$\mathbb{P}(Y(t) \le x) = \mathbb{P}(Y_1(t) \le x)\mathbb{P}(Y_2(t) \le x) = F^{[t]}(x) \cdot F^{\langle t\rangle}(x) = F^t(x).$$

Both the processes Y_1 and Y_2 are non-decreasing and have right hand side continuous trajectories. $\square$

Proof of Proposition 5.3. Suppose that for some deterministic $c_p > 0$ we have convergence in distribution to some (necessarily non-negative) random variable Z stated in the proposition. Expressing this in terms of Laplace transforms (LTs) we will have

$$\mathbb{E}(e^{-tc_pN_p}) = G_p(e^{-tc_p}) \to \psi(t) \text{ as } p \to 0$$

for each $t > 0$, where $\psi(\cdot)$ is the LT of Z and $G_p(s)$ is the PGF of N_p given by the right-hand-side of (2.2) with $\alpha = p$. Consequently, we conclude that, for each $t > 0$, we would have

$$(1 - e^{-tc_p})^p \to 1 - \psi(t) \text{ as } p \to 0. \tag{6.3}$$

However, we have

$$(1 - e^{-tc_p})^p = \left(\frac{1 - e^{-tc_p}}{c_p} c_p\right)^p = \left(\frac{1 - e^{-tc_p}}{c_p}\right)^p c_p^p \tag{6.4}$$

and, since we must have $c_p \to 0$ as $p \to 0$, we would have

$$\left(\frac{1 - e^{-tc_p}}{c_p}\right)^p \to t^0 = 1 \text{ as } p \to 0.$$

In view of this, along with (6.3)-(6.4), we conclude that, for each $t > 0$, we must have the convergence

$$c_p^p \to 1 - \psi(t) \text{ as } p \to 0,$$

which can not hold unless the right-hand-side is independent on t, which is the case only if $\mathbb{P}(Z = 0) = 1$. This concludes the proof. $\square$

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