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NICOLA LOPERFIDO, UNIVERSITÀ DEGLI STUDI DI URBINO ”CARLO BO”
STEPAN MAZUR, LUND UNIVERSITY
KRZYSZTOF PODGÓRSKI, LUND UNIVERSITY

Third Cumulant for Multivariate Aggregate Claim Models

Nicola Loperfido[a], Stepan Mazur[b] and Krzysztof Podgórski[b] [1]

[a] *Dipartimento di Economia, Società e Politica, Università degli Studi di Urbino "Carlo Bo", Via Saffi 42, 61029 Urbino (PU), Italy*

[b] *Department of Statistics, Lund University, Tycho Brahes väg 1, SE-22007 Lund, Sweden*

Abstract

The third moment cumulant for the aggregated multivariate claims is considered. A formula is presented for the general case when the aggregating variable is independent of the multivariate claims. It is discussed how this result can be used to obtain a formula for the third cumulant for a classical model of multivariate claims. Two important special cases are considered. In the first one, multivariate skewed normal claims are considered and aggregated by a Poisson variable. The second case is dealing with multivariate asymmetric generalized Laplace and aggregation is made by a negative binomial variable. Due to the invariance property the latter case can be derived directly leading to the identity involving the cumulant of the claims and the aggregated claims. There is a well established relation between asymmetric Laplace motion and negative binomial process that corresponds to the invariance principle of the aggregating claims for the generalized asymmetric Laplace distribution. We explore this relation and provide multivariate continuous time version of the results.

Keywords: Third cumulant, multivariate aggregate claim, skew-normal, Laplace motion

[1] Corresponding author. E-mail address: stepan.mazur@stat.lu.se. The authors appreciate the financial support of the Swedish Research Council Grant Dnr: 2013-5180 and Riksbankens Jubileumsfond Grant Dnr: P13-1024:1

1 Introduction

The joint aggregated claim models for the multivariate claims is of the fundamental importance for risk assessment in actuarial sciences. In fact, insurance claims usually arrive from contracts that are insuring against accidents of different nature but dependent in their occurrences. For example, it is natural to expect that weather conditions affect forest fires and draught in farms and thus inducing dependence for claim size arriving from such accidents. Dependencies can be present in the severities of the claims of different type as well as in the numbers of claims of different types occurring during particular types of accidents. Actuaries need tools that would facilitate the analysis of models that account for multivariate dependence structure. In the pioneering paper of Cummins and Wiltbank (1983), it was pointed out that the dependence in the aggregated multivariate claim sizes (severities) can occur at three different levels: dependencies between numbers (frequencies) of different types of accident resulting in multivariate losses (number of forest fires and farm droughts are naturally correlated through weather conditions); the aggregated numbers of different types of claims within a particular accident type (number of injury claims and number of damage claims in vehicle accidents); a joint claim size distribution for the multivariate claims corresponding to each type of accident (in a car accident, the damage to the vehicle and severities of injuries of individual involved in the accident). Following this seminal work, a number of multivariate extensions of the standard aggregated one-dimensional Cramer-Andersen collective risk model have been discussed in the literature, see Ivanova and Khokhlov (2007), Ambagaspitiya (1999), and Ren (2012) and the references therein. Despite of the importance of the models, understanding the theoretical properties of these models is lagging as compared for the theory of one dimensional claims.

The skewness of the records has a major impact on the risk assessment resulting from the aggregated claims. For the multivariate distributions, the third moment cumulant can be viewed as multivariate generalization of the skewness that plays the quite important role, see Kollo and von Rosen (2005), De Luca and Loperfido (2015) and others. For example, one can approximate the density function of the aggrageted claims using its cumulants (see McCullagh (1987), Barndorff-Nielsen and Cox (1989), Kollo and von Rosen (1998)). However, until now, the workable formula for the third cumulant of the aggregated claims was not known. The main contribution of this paper lies in deriving a general formula for the third cumulant that can be applied to a number of practically relevant models of aggregated multivariate claims.

In a generic form, the aggregate claims in the multivariate case describe aggregation of sizes of claims from a number of types of accident each carrying a certain number of types of claims (Cummins and Wiltbank (1983)), while Anastasiadis and Chukova (2012) provides a recent overview of the developments. More precisely in this model it is assumed that we deal with a certain number, say B, of different types of claims and their aggregated severities (sizes) per accident constitute a random vector $\mathbf{X}_a = (X_{a1}, \ldots, X_{aB})$, where a is referring to a type of the accident, $a = 1, \ldots, A$. Since in each accident there can be a certain random number K_{ab} of the incidents of claims of the type b, X_{ab}'s are, in fact, aggregations of individual claims, i.e.

$$X_{ab} = X_{ab1} + \cdots + X_{abK_{ab}} = \sum_{j=1}^{K_{ab}} X_{abj}, \ \ b = 1, \ldots, B.$$

For example, if a refers to the automobile accident, and b to bodily injury liability, then if in an accident there is $K_{ab} = 3$ injured persons, then X_{abj} refers to the claim value by the jth injured person, $j = 1, 2, 3$. Generally, it is assumed that for different a and different instances i of the ath accident, the aggregated claims $\mathbf{X}_{ai}$ are independent.

Typically, it is assumed that given fixed b, X_{abji}'s are independent when different a (accidents), j (individual claims) and i (instances of accidents of the type a) are considered together with independence of the claim numbers K_{abi}. Consequently, one can assume independence between aggregated claims for accidents of type a:

$$\mathbf{S}_a = \mathbf{X}_{a1} + \mathbf{X}_{a2} + ... + \mathbf{X}_{aN_a}, \quad a = 1, \ldots, A,$$

where A is the number of types of accidents, N_a is the random number of occurrences of the a-type accidents and $\mathbf{X}_{ai}$ is the vector of sizes of different types of claims in the ith occurrence of an a type accident. The total aggregated claim sizes in a certain time period writes as

$$\mathbf{S} = \mathbf{S}_1 + \cdots + \mathbf{S}_A. \tag{1}$$

The coordinates of $\mathbf{S} = (S_1, \ldots, S_B)$ are the aggregated claim sizes of B claim types. It is clear that these coordinates are dependent and understanding the structure of this dependence is of great interest for an actuary. The third cumulant allows to summarize the skewness of the multivariate structure of claims and results this work provides tools for effective evaluation of this important multivariate characteristic.

In the next section, we present mathematical properties of the third cumulants for linear combination of multivariate vectors that allow, under the assumption of independence of $\mathbf{S}_a$'s, to express the third cumulant of $\mathbf{S}$ in terms of the third cumulants of $\mathbf{S}_a$, see Corollary 1 of Section 2. In fact, it would be also possible to find the third cumulant without the independence but this would require cross-cumulants as seen in Proposition 2, Section 2. However, we do not explore this direction in the present work. We conclude this section with a discussion of validity of independence between $\mathbf{S}_a$'s.

It is argued in the literature that the assumption of independence is on some occasions not supported in reality. For example, the fire of a house can be triggered by severe hot weather condition which can also induce medical conditions among elders. Thus under these conditions home insurance claims will stochastically dependent on claims following from the health insurance contracts and different types accidents do not necessarily occur independently of each other. One way to model such a dependence is to consider finer partition of the type of events ('accidents') that would treat jointly the claims that follows from the similar circumstances (claims during extreme temperature incidents) but separating the types of accidents do not occur simultaneously (claims under regular weather conditions). This allows multivariate grouping of the claim sizes into the aggregated claim models $\mathbf{S}_a$ with the independent components for different a's. From such an 'independent' type of accident model one can obtain the dependent one by a choice of a linear transformation of the independent component. This interpretation of dependence between different types of accidents has been explored, for example, in Ivanova and Khokhlov (2007). Since the third cumulant of a linear transformation of a random vector can be expressed using the cumulants of the independent components, see Proposition 2, Section 2, we conclude that for the purpose of this paper the assumption of independence can be maintained and it is enough to discuss properties for individual $\mathbf{S}_a$.

The main part of the paper is Section 3, where a general result on the third cumulant for the total aggregated claim with independent components is presented. From the above discussion, it follows that in principle it also applicable to produce the third cumulant for a general Cramer-Andersen model, although the explicit formula would require further specifications of the model which is not the subject of this paper.

2 The third cumulant – definition and properties

Let $\boldsymbol{\mu} \in \mathbb{R}^d$ be the mean vector and $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$ be the covariance matrix of a d-dimensional random vector $\mathbf{X}$. Then the third cumulant of $\mathbf{X}$ is a third order tensor, often represented as the $d^2 \times d$ matrix,

$$\kappa_3(\mathbf{X}) = E\left\{(\mathbf{X}-\boldsymbol{\mu}) \otimes (\mathbf{X}-\boldsymbol{\mu}) \otimes (\mathbf{X}-\boldsymbol{\mu})^T\right\},$$

where $\otimes$ denotes the Kronecker (tensor) product of matrices. This a natural third order concept that parallels the mean $\kappa_1(\mathbf{X})$ and covariance $\kappa_2(\mathbf{X})$:

$$\begin{aligned}
\kappa_1(\mathbf{X}) &= E(\mathbf{X}),\\
\kappa_2(\mathbf{X}) &= E\left\{(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T\right\} = E\left\{(\mathbf{X}-\boldsymbol{\mu}) \otimes (\mathbf{X}-\boldsymbol{\mu})^T\right\}.
\end{aligned}$$

There are a number of basic properties of the third cumulant that can be found in Kollo and von Rosen (2005), De Luca and Loperfido (2015) and others. Here we will list only these that are important for the results of this paper.

In what follows the third cumulant is treated as a $d^2 \times d$ matrix and the linear transformation $\mathbf{Y} = \mathbf{AX}$, where $\mathbf{A}$ is a $r \times d$ real matrix has the third cumulant that is given by

$$\kappa_3(\mathbf{Y}) = (\mathbf{A} \otimes \mathbf{A})\kappa_3(\mathbf{X})\mathbf{A}^T, \tag{2}$$

where $\mathbf{A} \otimes \mathbf{A}$ is treated as a $r^2 \times d^2$ matrix (see Kollo and von Rosen (2005), p.190).

The following property is particularly useful when the effect of linear dependencies has been removed by considering the standardized random vector $\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$, where $\mathbf{\Sigma}^{-1/2}$ is the (unique) symmetric and positive definite square-root matrix of the inverse of the covariance matrix $\mathbf{\Sigma}$, which is assumed to be non-singular. The third standardized cumulant of $\mathbf{X}$ is the third cumulant of $\mathbf{Z}$. It is often denoted by $\overline{\kappa}_3(\mathbf{X})$, and is related to $\kappa_3(\mathbf{Z})$ via the equality

$$\kappa_3(\mathbf{Z}) = (\mathbf{\Sigma}^{-1/2} \otimes \mathbf{\Sigma}^{-1/2})\kappa_3(\mathbf{X})\mathbf{\Sigma}^{-1/2} = \overline{\kappa}_3(\mathbf{X}).$$

The skewness of a random variable X satisfying $E\left(|X|^3\right) < +\infty$ is often measured by its third standardized cumulant

$$\gamma_1(X) = E\left[\frac{(X-\mu)^3}{\sigma^3}\right],$$

where μ and σ^2 are the mean and the variance of X, respectively. The squared third standardized cumulant $\beta_1(X) = \gamma_1^2(X)$, known as Pearson's skewness, is also used.

The third cumulant of a d-dimensional random vector is a $d^2 \times d$ matrix with at most $d(d+1)(d+2)/6$ distinct elements. Since their number grows very quickly with the vector's dimension, it is convenient to summarize the skewness of the random vector itself with a scalar function of the third standardized cumulant, as for example Mardia's skewness, partial skewness and directional skewness. They have been mainly used for testing multivariate normality, are invariant with respect to one-to-one affine transformations and reduce to Pearson's skewness in the univariate case. Loperfido (2015b) reviews their main properties and investigates their mutual connections.

Mardia (1970) defined the skewness of a random vector $\mathbf{X}$ as

$$\beta_{1,d}^M(\mathbf{X}) = E\left[\left(\mathbf{Z}^T\mathbf{W}\right)^3\right],$$

where $\mathbf{W} = \mathbf{\Sigma}^{-1/2}(\mathbf{Y} - \boldsymbol{\mu})$, $\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$, while $\mathbf{X}$ and $\mathbf{Y}$ are two d-dimensional, independent and identically distributed random vectors with mean $\boldsymbol{\mu}$ and variance $\mathbf{\Sigma}$. Mardia's skewness can be defined as the squared Forbenius norm of the third standardized cumulant as

$$\beta_{1,d}^M(\mathbf{X}) = ||\overline{\kappa}_3(\mathbf{X})||^2.$$

Mardia's skewness is by far the most popular measure of multivariate skewness. Its statistical applications include multivariate normality testing (see Mardia (1970)) and assessment of robustness of MANOVA statistics (see Davis (1980)).

Another measure of multivariate skewness is

$$\beta_{1,d}^{P}(\mathbf{X}) = E\left(\mathbf{Z}^T\mathbf{Z}\mathbf{Z}^T\mathbf{W}\mathbf{W}^T\mathbf{W}\right),$$

where $\mathbf{Z}$ and $\mathbf{W}$ are the same as above. It has been independently proposed by several authors (Davis (1980), Isogai (1983), Mòri et al. (1993)). The name partial skewness reminds that $\beta_{1,d}^{P}(\mathbf{X})$ does not depend on the cross-product moment $E(z_i z_j z_k)$ when i, j, k differ from each other. The partial skewness is far less popular than Mardia's skewness. Like the latter measure, however, it has been applied to multivariate normality testing (see Henze (1997)) and to the assessment of the robustness of MANOVA statistics (see Davis (1980)). Moreover, Loperfido (2015a) showed that the partial skewness can be obtained as

$$\beta_{1,d}^{P}(\mathbf{X}) = || \left[\overline{\kappa}_3(\mathbf{X})\right]^T \mathbf{I}^V ||^2,$$

where $\mathbf{I}^V$ denote the vector obtained by stacking the columns of the identity matrix $\mathbf{I}$ on top of each other.

Malkovich and Afifi (1973) defined the multivariate skewness of a random vector $\mathbf{X}$ as the maximum value $\beta_{1,d}^{D}(\mathbf{X})$ attainable by $\beta_1\left(\mathbf{C}^T\mathbf{X}\right)$, where $\mathbf{C}$ is a nonnull, d-dimensional real vector and $\beta_1(Y)$ is the squared third standardized moment of the random variable Y. The name directional skewness reminds that $\beta_{1,d}^{D}(\mathbf{X})$ is the maximum attainable skewness by a projection of the random vector $\mathbf{X}$ onto a direction. Statistical applications of directional skewness include normality testing (see Malkovich and Afifi (1973)), point estimation (see Loperfido (2010)), projection pursuit and cluster analysis (see Loperfido (2015b)).

We conclude this section with the relation between the third cumulant of a vector $\mathbf{X}$ that is a matrix linear combination of two vectors $\mathbf{Y}$ and $\mathbf{Z}$ with dimensions a and b, respectively. Namely, we consider

$$\mathbf{X} = \mathbf{A}\mathbf{Y} + \mathbf{B}\mathbf{Z}, \tag{3}$$

where $\mathbf{A}$ and $\mathbf{B}$ are an arbitrary matrices with sizes $d \times a$ and $d \times b$, respectively.

Before formulating the result we list some basic properties of the tensor product. We refer to Harville (2008), and Mardia et al. (1979) for more details. In all properties, we assume that the dimensions of the matrices are such that the operations on them are well defined.

For arbitrary vectors $\mathbf{F}$ and $\mathbf{G}$ treated as column matrices

$$\mathbf{F} \otimes \mathbf{G}^T = \mathbf{G}^T \otimes \mathbf{F} = \mathbf{F}\mathbf{G}^T. \tag{4}$$

For matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ we have the following linearity properties

$$\begin{aligned} (\mathbf{A}+\mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}, \\ \mathbf{C} \otimes (\mathbf{A}+\mathbf{B}) = \mathbf{C} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{B}, \end{aligned} \tag{5}$$

and the following commutative-associative property

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}. \tag{6}$$

We have the following result that generalizes (2).

Proposition 1. *Let the cross cumulants of the third order between to vector variables* $\mathbf{Z}$ *and* $\mathbf{Y}$ *be defined through*

$$\begin{aligned} \kappa_{21}(\mathbf{Y},\mathbf{Z}) = E\left\{(\mathbf{Y}-\boldsymbol{\mu}_\mathbf{Y}) \otimes (\mathbf{Y}-\boldsymbol{\mu}_\mathbf{Y}) \otimes (\mathbf{Z}-\boldsymbol{\mu}_\mathbf{Z})^T\right\}, \\ \kappa_{12}(\mathbf{Y},\mathbf{Z}) = E\left\{(\mathbf{Y}-\boldsymbol{\mu}_\mathbf{Y}) \otimes (\mathbf{Z}-\boldsymbol{\mu}_\mathbf{Z}) \otimes (\mathbf{Z}-\boldsymbol{\mu}_\mathbf{Z})^T\right\}, \end{aligned}$$

whenever all the required moments exist. Then for matrices $\mathbf{A}$ *and* $\mathbf{B}$ *we have*

$$\begin{aligned} \kappa_{21}(\mathbf{AY},\mathbf{BZ}) &= (\mathbf{A} \otimes \mathbf{A})\kappa_{21}(\mathbf{Y},\mathbf{Z})\mathbf{B}^T, \\ \kappa_{12}(\mathbf{AY},\mathbf{BZ}) &= (\mathbf{A} \otimes \mathbf{B})\kappa_{12}(\mathbf{Y},\mathbf{Z})\mathbf{B}^T. \end{aligned}$$

Proof. Without the loss of the generality, we assume that $\boldsymbol{\mu}_\mathbf{Y}$ and $\boldsymbol{\mu}_\mathbf{Z}$ are equal to zero. By definition

$$\kappa_{21}(\mathbf{AY},\mathbf{BZ}) \quad = E\left\{(\mathbf{AY}) \otimes (\mathbf{AY}) \otimes (\mathbf{BZ})^T\right\}.$$

Applying (6) to the last formula we have that

$$\kappa_{21}(\mathbf{AY},\mathbf{BZ}) \quad = E\left\{(\mathbf{A} \otimes \mathbf{A})(\mathbf{Y} \otimes \mathbf{Y}) \otimes (\mathbf{Z}^T\mathbf{B}^T)\right\}.$$

Finally, linear properties of the expected value imply

$$\kappa_{21}(\mathbf{AY},\mathbf{BZ}) \quad = (\mathbf{A} \otimes \mathbf{A})E\left\{\mathbf{Y} \otimes \mathbf{Y} \otimes \mathbf{Z}^T\right\}\mathbf{B}^T.$$

Using the same properties for $\kappa_{12}(\mathbf{AY},\mathbf{BZ})$ we obtain the statement of the proposition. □

Before reporting the result on the third cumulant for linear transformation of vectors, we note that if $\mathbf{X}$ and $\mathbf{Y}$ are independent, then

$$E(\mathbf{X}\otimes\mathbf{Y}) = E\mathbf{X}\otimes E\mathbf{Y}.$$

In particular all third order cross cumulants are zero.

Proposition 2. *For* $\mathbf{X}$ *given in (3) we have*

$$\begin{aligned}\kappa_3(\mathbf{X}) &= (\mathbf{A}\otimes\mathbf{A})\kappa_3(\mathbf{Y})\mathbf{A}^T + (\mathbf{B}\otimes\mathbf{B})\kappa_3(\mathbf{Z})\mathbf{B}^T \\ &+ 2\left((\mathbf{A}\otimes\mathbf{A})\kappa_{21}(\mathbf{Y},\mathbf{Z})\mathbf{B}^T + (\mathbf{B}\otimes\mathbf{B})\kappa_{21}(\mathbf{Z},\mathbf{Y})\mathbf{A}^T\right) \\ &+ (\mathbf{B}\otimes\mathbf{A})\kappa_{12}(\mathbf{Z},\mathbf{Y})\mathbf{A}^T + (\mathbf{A}\otimes\mathbf{B})\kappa_{12}(\mathbf{Y},\mathbf{Z})\mathbf{B}^T.\end{aligned}$$

If we assume additionally that $\mathbf{Y}$ *and* $\mathbf{Z}$ *are independent, then*

$$\kappa_3(\mathbf{X}) = (\mathbf{A}\otimes\mathbf{A})\kappa_3(\mathbf{Y})\mathbf{A}^T + (\mathbf{B}\otimes\mathbf{B})\kappa_3(\mathbf{Z})\mathbf{B}^T.$$

Proof. We can assume without losing generality that $\boldsymbol{\mu}_\mathbf{X}$, $\boldsymbol{\mu}_\mathbf{Y}$, and $\boldsymbol{\mu}_\mathbf{Z}$ are all zero. By the linearity properties (5) of the tensor product

$$\begin{aligned}\kappa_3(\mathbf{X}) &= E\left\{(\mathbf{AY}+\mathbf{BZ})\otimes(\mathbf{AY}+\mathbf{BZ})\otimes(\mathbf{AY}+\mathbf{BZ})^T\right\} \\ &= E\left\{(\mathbf{AY})\otimes(\mathbf{AY})\otimes(\mathbf{AY})^T\right\} + E\left\{(\mathbf{BZ})\otimes(\mathbf{BZ})\otimes(\mathbf{BZ})^T\right\} \\ &+ E\left\{(\mathbf{AY})\otimes(\mathbf{BZ})\otimes(\mathbf{AY})^T\right\} + E\left\{(\mathbf{BZ})\otimes(\mathbf{AY})\otimes(\mathbf{AY})^T\right\} \\ &+ E\left\{(\mathbf{BZ})\otimes(\mathbf{BZ})\otimes(\mathbf{AY})^T\right\} + E\left\{(\mathbf{AY})\otimes(\mathbf{AY})\otimes(\mathbf{BZ})^T\right\} \\ &+ E\left\{(\mathbf{AY})\otimes(\mathbf{BZ})\otimes(\mathbf{BZ})^T\right\} + E\left\{(\mathbf{BZ})\otimes(\mathbf{AY})\otimes(\mathbf{BZ})^T\right\}.\end{aligned}$$

By (2) and (4), we get

$$\begin{aligned}\kappa_3(\mathbf{X}) &= (\mathbf{A}\otimes\mathbf{A})\kappa_3(\mathbf{Y})\mathbf{A}^T + (\mathbf{B}\otimes\mathbf{B})\kappa_3(\mathbf{Z})\mathbf{B}^T \\ &+ 2E\left\{(\mathbf{AY})\otimes(\mathbf{AY})\otimes(\mathbf{BZ})^T\right\} + 2E\left\{(\mathbf{BZ})\otimes(\mathbf{BZ})\otimes(\mathbf{AY})^T\right\} \\ &+ E\left\{(\mathbf{BZ})\otimes(\mathbf{AY})\otimes(\mathbf{AY})^T\right\} + E\left\{(\mathbf{AY})\otimes(\mathbf{BZ})\otimes(\mathbf{BZ})^T\right\}.\end{aligned}$$

Applying Proposition 1 to the last formula we get the first statement of the proposition.

The case when $\mathbf{Y}$ and $\mathbf{Z}$ are independently distributed, follows easily since cross cumulants vanish. □

From Proposition 2 we get the third cumulant of a random vector $\mathbf{X}$ is made of entries of two independent vectors $\mathbf{Y}$ and $\mathbf{Z}$.

Corollary 1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *have the following forms*

$$\mathbf{A} = \begin{bmatrix}\mathbf{I}_a \\ \mathbf{0}\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}\mathbf{0} \\ \mathbf{I}_b\end{bmatrix},$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and $\mathbf{0}$ stand for zero matrices with dimensions not represented in the notation and chosen so that $\mathbf{A}$ and $\mathbf{B}$ have sizes $d \times a$ and $d \times b$, respectively. Then, $\kappa_3(\mathbf{X})$ for a case when $\mathbf{Y}$ and $\mathbf{Z}$ are independently distributed can be expressed in the block matrix form

$$\kappa_3(\mathbf{X}) = \begin{bmatrix} (\mathbf{I}_a \otimes \mathbf{A})\, \kappa_3(\mathbf{Y}) & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}_b \otimes \mathbf{B})\, \kappa_3(\mathbf{Z}) \end{bmatrix}$$

and we have

$$\begin{bmatrix} \kappa_3(\mathbf{Y}) & \mathbf{0} \\ \mathbf{0} & \kappa_3(\mathbf{Z}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_a \otimes \mathbf{A}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_b \otimes \mathbf{B}^T \end{bmatrix} \kappa_3(\mathbf{X}),$$

where the last result follows from the identity

$$(\mathbf{I}_a \otimes \mathbf{A}) \left(\mathbf{I}_a \otimes \mathbf{A}^T\right) = \mathbf{I}_{a^2}.$$

3 The third cumulant for aggregated claims

We have seen that from the above properties we can reduce the problem of evaluating the third cumulant of multivariate claims to the aggregate claims of the following form

$$\mathbf{S} = \mathbf{X}_1 + \mathbf{X}_2 + ... + \mathbf{X}_N, \tag{7}$$

where $\mathbf{X}_1, \ldots, \mathbf{X}_N$ are identically and independently distributed d-dimensional claim sizes random vectors and N is the univariate claim count random variable which is independent of the $\mathbf{X}_1, \ldots, \mathbf{X}_N$. We note that $\mathbf{X}_i$'s and N should have the finite moments up to the third order.

3.1 General formula

Next, we consider the third cumulant of the aggregate claims for the general case when the aggregating variable is independent of the claims. This result is presented in the following theorem.

Theorem 1. *Let N be a nonnegative random variate with finite the third moment. Moreover, let $\mathbf{X}_1, \ldots, \mathbf{X}_N$ be identically distributed random vectors with a finite third moment, independent of each other as well as independent of N. Then the third cumulant of the multivariate aggregate claims is given by*

$$\kappa_3(\mathbf{S}) = \nu_1 \boldsymbol{\xi}_3 + \nu_2 \left(\boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2^V \boldsymbol{\xi}_1^T + \boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_2\right) + \nu_3 \boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1^T \otimes \boldsymbol{\xi}_1,$$

where ν_j and $\boldsymbol{\xi}_j$ $(j = 1, 2, 3)$ denote the $j-th$ cumulants of N and $\mathbf{X}_i$, respectively, and $\boldsymbol{\xi}_2^V$ denote the vector obtained by stacking the columns of the $\boldsymbol{\xi}_2$ on top of each other.

From Theorem 1 we get that the third cumulant of multivariate aggregate claims is expressed as the function the first three cumulants of the random variable N and the random vector $\mathbf{X}_i$. In the next corollary, we present the third cumulant of $\mathbf{S}$ using non-centered and centered moments of the random variable N and the random vector $\mathbf{X}_i$. We note that Corollary 2 follows directly from Theorem 1.

Corollary 2. *Let N be a nonnegative random variate with finite third moment. Moreover, let $\mathbf{X}_1, \ldots, \mathbf{X}_N$ be identically distributed random vectors with finite third moment, independent of each other as well as independent of N. Then the third cumulant of the multivariate aggregate claims is given by*

$$\kappa_3(\mathbf{S}) = \mu_{N,1} \cdot \overline{\mu}_{\mathbf{X}_i,3} + 3\mu_{\mathbf{X}_i,1} \cdot \overline{\mu}_{\mathbf{X}_i,2} \cdot \overline{\mu}_{N,2} + \mu^3_{\mathbf{X}_i,1} \cdot \overline{\mu}_{N,3},$$

where $\mu_{Y,i}$ and $\overline{\mu}_{Y,i}$ are the $i-th$ noncentered and centered moment, respectively.

Remark 1. *It is noted that Corollary 2 is a well known result, and can be found, for example, in Cummins and Wiltbank (1983), or Panjer and Willmot (1992). Also, univariate cumulants, included the third and the fourth, are considered in Chaubey et al. (1998) and Sundt et al. (1998).*

In the following subsection we present application of the result to some concrete aggregated claims model.

3.2 The Poisson-skewed normal aggregated claims model

Here, we consider the multivariate aggregate claims with multivariate claim sizes which have the skew-normal distribution and the claim count which has a Poisson distribution.

We remind that the distribution of a d-dimensional random vector $\mathbf{y}$ is a multivariate skew-normal (SN, henceforth) with scale parameter $\boldsymbol{\Omega} \in \mathbb{R}^d \times \mathbb{R}^d$ and shape parameter $\boldsymbol{\alpha} \in \mathbb{R}^d$, i.e. $\mathbf{X}_i \sim \mathcal{SN}_d(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, if its density function (see Azzalini and Dalla Valle (1996)) is given by

$$f(\mathbf{y}; \boldsymbol{\Omega}, \boldsymbol{\alpha}) = 2\phi_d(\mathbf{y}; \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^T \mathbf{y}),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable and $\phi_d(\mathbf{y}; \boldsymbol{\Omega})$ is the density function of a d-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Omega}$. The importance of SN distribution in finance and actuarial science is well described by Adcock et al. (2015).

In the following theorem, we derive the third cumulant of the aggregate cliams under assumptions that claims have a SN distribution and the aggregation is made by Poisson distribution. Moreover, we derive its third standardized cumulant, Mardia's, partial and directional skewness.

Theorem 2. *Let N be a Poisson random variable with mean λ. Moreover, let $\mathbf{X}_1, \ldots, \mathbf{X}_N$ be skew-normal random vectors with scale matrix $\boldsymbol{\Omega}$ and shape vector $\boldsymbol{\alpha}$, independent of each other as well as of N. Additionally, let $\boldsymbol{\delta} = \dfrac{\boldsymbol{\Omega\alpha}}{\sqrt{1+\boldsymbol{\alpha}^T\boldsymbol{\Omega\alpha}}}$, $\boldsymbol{\eta} = \boldsymbol{\Omega}^{-1/2}\boldsymbol{\delta}$, and $q = \boldsymbol{\alpha}^T\boldsymbol{\Omega\alpha}/\left(1+\boldsymbol{\alpha}^T\boldsymbol{\Omega\alpha}\right)$. Then for the multivariate aggregated claim model $\mathbf{S}$ given in (7) we have*

(a) the third cumulant

$$\kappa_3(\mathbf{S}) = \lambda\sqrt{\frac{2}{\pi}}\left(\boldsymbol{\Omega}\otimes\boldsymbol{\delta} + \boldsymbol{\Omega}^V\boldsymbol{\delta}^T + \boldsymbol{\delta}\otimes\boldsymbol{\Omega} - \boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}\right);$$

(b) the third standardized cumulant

$$\overline{\kappa}_3(\mathbf{S}) = \sqrt{\frac{2}{\lambda\pi}}\left(\mathbf{I}\otimes\boldsymbol{\eta} + \mathbf{I}^V\boldsymbol{\eta}^T + \boldsymbol{\eta}\otimes\mathbf{I} - \boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right);$$

(c) Mardia's skewness, partial skewness and directional skewness, respectively

$$\beta^M_{1,d}(\mathbf{S}) = \frac{2q}{\lambda\pi}\left[3\,(d-1) + (q-3)^2\right], \quad \beta^P_{1,d}(\mathbf{S}) = \frac{2q}{\lambda\pi}\,(d+2+q)^2, \quad \beta^D_{1,d}(\mathbf{S}) = \frac{2q\,(3-q)^2}{\lambda\pi}.$$

3.3 Generalized multivariate Laplace claim sizes

Another important multivariate aggregated claims model can be obtained by considering in (7) asymmetric multivariate Laplace distribution for the claim sizes $\mathbf{X}_i$. Recall, that a random vector $\mathbf{Y} \in \mathbb{R}^d$ is said to have a multivariate generalized asymmetric Laplace distribution (GAL) if its characteristic function is given by

$$\varphi_{\mathbf{Y}}(\mathbf{t}) = \left(\frac{1}{1+\frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t} - i\boldsymbol{\mu}^T\mathbf{t}}\right)^s, \qquad \mathbf{t}\in\mathbb{R}^d,$$

where $s > 0$, $\boldsymbol{\mu} \in \mathbb{R}^d$, and $\boldsymbol{\Sigma}$ is a $d \times d$ non-negative definite symmetric matrix. This distribution is denoted by $GAL_d(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s)$. Also, the GAL random vector $\mathbf{Y}$ has the following stochastic representation

$$\mathbf{Y} \overset{d}{=} \boldsymbol{\mu}\Gamma + \sqrt{\Gamma}\mathbf{X} \overset{d}{=} \mathbf{B}_d(\Gamma), \tag{8}$$

where Γ has a standard gamma distribution with shape parameter s, $\mathbf{X} \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma})$, while $\mathbf{B}_d(t)$ is the multivariate Brownian motion with the drift $\boldsymbol{\mu}$ and the covariance matrix parameter $\boldsymbol{\Sigma}$. The symbol $\overset{d}{=}$ denotes the equality in distribution. Stochastic representation (8) shows that GAL distributions are location-scale mixtures of normal distributions.

For our discussion we need the third cumulant for GAL distributions. The result has been derived in Hürlimann (2013). Alternatively, one can apply a general result about the third cumulant of the process that is obtained by subordination of a Brownian motion with drift to a random time change. We consider a non-negative random variable Γ independent of a Brownian motion $\mathbf{B}_d$ and consider

$$\mathbf{Y} = \mathbf{B}_d(\Gamma). \tag{9}$$

Theorem 3. *Let $\mathbf{B}_d$ be a d-variate Brownian motion with drift $\boldsymbol{\mu} \in \mathrm{R}^d$ and covariance matrix $\boldsymbol{\Sigma} \in \mathrm{R}^{d\times d}$. Then the third cumulant of $\mathbf{Y}$ in (9) is given by*

$$\kappa_3(\mathbf{Y}) \quad = \quad \zeta_2 \cdot (\boldsymbol{\Sigma}^V \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}) + \zeta_3 \cdot \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T,$$

where ζ_2 and ζ_3 denote the second and third cumulants of Γ, respectively.

Proof. Let us note that the mean vector of $\mathbf{Y}$ is $\boldsymbol{\mu}_{\mathbf{Y}} = E(\Gamma)\boldsymbol{\mu}$. If we consider the Brownian motion without drift $\mathbf{B}_d^0(t) = \mathbf{B}_d(t) - \boldsymbol{\mu}t$ and the centered variable $\Gamma_0 = \Gamma - E(\Gamma)$, then from the definition of the third cumulant we have

$$\begin{aligned}
\kappa_3(\mathbf{Y}) \quad &= \quad E\left[(\mathbf{B}_d^0(\Gamma) + \boldsymbol{\mu}\Gamma_0) \otimes (\mathbf{B}_d^0(\Gamma) + \boldsymbol{\mu}\Gamma_0) \otimes (\mathbf{B}_d^0(\Gamma) + \boldsymbol{\mu}\Gamma_0)^T\right] \\
&= \quad \kappa_3(\mathbf{B}_d^0(\Gamma)) + E\left(\mathbf{B}_d^0(\Gamma) \otimes \mathbf{B}_d^0(\Gamma) \otimes \Gamma_0\boldsymbol{\mu}^T\right) \\
&\quad\quad + E\left(\mathbf{B}_d^0(\Gamma) \otimes \Gamma_0\boldsymbol{\mu} \otimes \mathbf{B}_d^0(\Gamma)^T\right) + E\left(\mathbf{B}_d^0(\Gamma) \otimes \Gamma_0\boldsymbol{\mu} \otimes \Gamma_0\boldsymbol{\mu}^T\right) \\
&\quad\quad + E\left(\Gamma_0\boldsymbol{\mu} \otimes \mathbf{B}_d^0(\Gamma) \otimes \mathbf{B}_d^0(\Gamma)^T\right) + E\left(\Gamma_0\boldsymbol{\mu} \otimes \mathbf{B}_d^0(\Gamma) \otimes \Gamma_0\boldsymbol{\mu}^T\right) \\
&\quad\quad + E\left(\Gamma_0\boldsymbol{\mu} \otimes \Gamma_0\boldsymbol{\mu} \otimes \mathbf{B}_d^0(\Gamma)^T\right) + E\left(\Gamma_0\boldsymbol{\mu} \otimes \Gamma_0\boldsymbol{\mu} \otimes \Gamma_0\boldsymbol{\mu}^T\right)
\end{aligned}$$

Since $\mathbf{B}_d^0(\Gamma)$ is symmetric its third cumulant is zero and thus

$$\begin{aligned}
\kappa_3(\mathbf{Y}) = {} & E\left(\Gamma_0 E\left(\mathbf{B}_d^0(\Gamma) \otimes \mathbf{B}_d^0(\Gamma)|\Gamma\right) \otimes \boldsymbol{\mu}^T\right) + E\left(\Gamma_0 E\left(\mathbf{B}_d^0(\Gamma) \otimes \boldsymbol{\mu} \otimes \mathbf{B}_d^0(\Gamma)^T|\Gamma\right)\right) \\
& + E\left(\Gamma_0\boldsymbol{\mu} \otimes E\left(\mathbf{B}_d^0(\Gamma) \otimes \mathbf{B}_d^0(\Gamma)^T|\Gamma\right)\right) + \zeta_3\ \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T \\
& + E\left(\Gamma_0^2\left(\boldsymbol{\mu} \otimes E\left(\mathbf{B}_d^0(\Gamma)|\Gamma\right) \otimes \boldsymbol{\mu}^T + E\left(\mathbf{B}_d^0(\Gamma)|\Gamma\right) \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes E\left(\mathbf{B}_d^0(\Gamma)^T|\Gamma\right)\right)\right).
\end{aligned}$$

Conditionally on Γ the vector $\mathbf{B}_d^0(\Gamma)$ is simply Gaussian and centered at zero with the covariance $\Gamma\boldsymbol{\Sigma}$, so the last line in the above vanishes and

$$\kappa_3(\mathbf{Y}) = E\left(\Gamma_0\Gamma\right)\left(\boldsymbol{\Sigma}^V\boldsymbol{\mu}^T + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma}\right) + \zeta_3\ \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T.$$

which proves the result. □

Corollary 3. *For the gamma distributed variable Γ with the shape parameter s and the scale equal to one, $\mathbf{Y}$ has the multivariate Laplace distribution and the formula for the* *third cumulant takes the form*

$$\kappa_3(\mathbf{Y}) = s \cdot (\mathbf{\Sigma}^V \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \mathbf{\Sigma} + \mathbf{\Sigma} \otimes \boldsymbol{\mu}) + 2s \cdot \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T$$

The model with GAL claim sizes can be compounded by various discrete valued random variables N. For example and similarly as before one can consider Poisson distributed N but it is more natural in this case to consider a negative binomial (NB) variable N_p for the compounding. It is due to some important invariance relations between the GAL and NB distributions. These properties are discussed next.

3.3.1 The negative binomial distributed number of claims

The negative binomial distribution arises naturally in the scheme of waiting for the failures in a sequence of Bernoulli trials. This happens to be infinitely divisible distribution and thus extend to a continuous time process $N_p(t)$, $t \geq 0$, see Kozubowski and Podgórski (2007) and Kozubowski and Podgórski (2009) for further details. Here just let us recall that $N_p(s)$ has a negative binomial distribution with parameters $p \in (0,1)$ and $s > 0$ given by

$$P(N_p(s) = k) = \frac{\Gamma(k)}{\Gamma(s)(k-s)!} p^s (1-p)^{k-s}, \quad k = s, s+1, s+2, \ldots.$$

The key property of the generalized Laplace distributions is that they are limiting distributions under the random summation schemes induced by N_p, with the probability of failure p being small (asymptotically convergent to zero). In fact, these distributions play a similar central role when one considers random number of terms in the summation as do the Gaussian distribution with nonrandom summations. This and their asymmetry/tail properties make the GAL distributions particularly suitable for application to compounding multivariate insurance claims. A exhaustive account of the properties for this class of distributions can be found in Kotz et al. (2001).

In particular, when the parameter $s = 1$, then $N_p = N_p(1)$ has geometric distribution and we have the limiting distribution of (normalized) geometric random sums

$$\mathbf{S}_p = \mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_{N_p},$$

as $p \to 0$ (and thus $N_p \to \infty$), is the multivariate Laplace distribution with $s = 1$, as long as $\mathbf{X}_i$ have second moment and are properly centered. This shows that approximately the geometric compounding always leads to the multivariate Laplace distribution, which stresses the importance of this distribution in theory of accidents and for modeling insurance claims, see Arbous and Kerrich (1951), Ferreri (1983), Lawless (1987) and Boucher et al. (2008).

In general, see Kozubowski et al. (2013), a similar result is true under negative binomial random compounding. Let $\mathbf{X}_i$ be identically and independently distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and let $N_p(s)$ be independent of $\mathbf{X}_i$'s. When $p \to 0$, we obtain that in the asymptotic sense compounding by the negative binomial process $N_p(s)$ of the centered $\tilde{\mathbf{X}}_i = \mathbf{X}_i + \mathbf{b}_p$, where $\mathbf{b}_p = \boldsymbol{\mu}(p^{1/2} - 1)$ always leads to a GAL distribution

$$\widetilde{\mathbf{S}} = a_p \sum_{i=1}^{N_p(s)} \tilde{\mathbf{X}}_i \sim GAL_d(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s), \tag{10}$$

where s is a positive integer and the normalizing constant $a_p = p^{1/2}$. Thus the third cumulative for multivariate GAL distribution presented in Corollary 3 can be viewed as an approximation of the third cumulant of the negative binomial compounding of any multivariate distribution. However, in one special case, this compounding yields exact result. We conclude this part with a discussion of this special but important case.

Consider a multivariate Laplace motion $\mathbf{L}_d(t)$, i.e. the Lévy process obtained by subordination of the Brownian motion $\mathbf{B}_d(t)$ with a gamma process $\Gamma(t)$. We consider here the standard gamma process for which $\Gamma(1)$ has the standard exponential distribution. As shown in Kozubowski and Podgórski (2007), if a negative binomial process $N_p(t)$ is independent of Laplace motion $\mathbf{L}_d$ the following invariance property of $\mathbf{L}_d(t)$ holds:

$$\mathbf{L}_d(t) \stackrel{d}{=} \sqrt{p} \cdot \mathbf{L}_d(N_p(t)).$$

Thus, in particular, if $\mathbf{X}$ has GAL distribution with parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and s, for a non-negative integer s, then we have

$$\mathbf{X} \stackrel{d}{=} \mathbf{L}_d(s) \stackrel{d}{=} \sqrt{p}\, \mathbf{L}_d(N_p(s)) = \sqrt{p} \sum_{n=1}^{N_p(s)} (\mathbf{L}_d(n) - \mathbf{L}_d(n-1)) \stackrel{d}{=} \sum_{n=1}^{N_p(s)} \mathbf{X}_n,$$

where $\mathbf{X}_n$ are iid having the multivariate asymmetric Laplace distribution with parameters $\sqrt{p}\boldsymbol{\mu}$, $p\boldsymbol{\Sigma}$, and $s = 1$. This leads to the following result.

Corollary 4. *The aggregated claims model (7) with the negative binomial distribution $N_p(t)$, $t \in \mathbb{N}$ of the number of claims and the multivariate asymmetric Laplace claim sizes with parameters $\sqrt{p}\boldsymbol{\mu}$, $p\boldsymbol{\Sigma}$, and $s = 1$ can be viewed as a stochastic time dependent model with $S(t)$, $t \in \mathbb{N}$ that adds new claims using increments of the negative binomial process $N_p(t)$. Additionally, let $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}$ and $r = \widetilde{\boldsymbol{\mu}}^T \widetilde{\boldsymbol{\mu}}$. Then for this model we have*

(a) the third cumulant

$$\kappa_3(\mathbf{S}(t)) = t \cdot (\boldsymbol{\Sigma}^V \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}) + 2t \cdot \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T;$$

(b) the third standardized cumulant

$$\overline{\kappa}_3(\mathbf{S}(t)) = t \cdot (\mathbf{I}^V \widetilde{\boldsymbol{\mu}}^T + \widetilde{\boldsymbol{\mu}} \otimes \mathbf{I} + \mathbf{I} \otimes \widetilde{\boldsymbol{\mu}}) + 2t \cdot \widetilde{\boldsymbol{\mu}} \otimes \widetilde{\boldsymbol{\mu}} \otimes \widetilde{\boldsymbol{\mu}}^T;$$

(c) Mardia's skewness, partial skewness and directional skewness, respectively

$$\begin{aligned} \beta_{1,d}^M(\mathbf{S}(t)) &= t^2 r \left[3(d-1) + 4\left(r + \frac{3}{2}\right)^2\right], \\ \beta_{1,d}^P(\mathbf{S}(t)) &= t^2 r(1+r+d)^2, \quad \beta_{1,d}^D(\mathbf{S}(t)) = t^2 r(3+2r)^2. \end{aligned}$$

The special case when the number of claims has geometric distribution is obtaining by taking $t = 1$. We note that the cumulant in this model does not depend on the choice of p.

3.3.2 The Poisson distributed number of claims

Another important model of the multivariate aggregate claims is obtained when the number of claim sizes follow the Poisson distribution, while the claim sizes have the multivariate GAL distribution. The importance of this model lies in its ability to obtain an interesting continuous time model as the subordination of the Laplace motion by a Poisson process

$$\mathbf{S}(t) = \mathbf{L}_d(\nu N(t)) = \sum_{k=1}^{N(t)} (\mathbf{L}_d(\nu k) - \mathbf{L}_d(\nu(k-1))) = \sum_{k=1}^{N(t)} \mathbf{X}_k, \tag{11}$$

where independent identically distributed claims $\mathbf{X}_k$ have a multivariate GAL distribution with parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and ν. Since the Laplace motion is obtained from the subordination from the Brownian motion having $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ for the parameters, we have

$$\mathbf{S}(t) = \mathbf{B}_d(\Gamma(\nu N(t))). \tag{12}$$

Consequently, the following result for the third moment of $\mathbf{S}(t)$ can be derived from either Theorem 1, or from Theorem 3.

Theorem 4. *For the model (11) or, equivalently, (12) we have the following formula for the third cumulant*

$$\kappa_3(\mathbf{S}(t)) = \nu^2 t \cdot (\boldsymbol{\Sigma}^V \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}) + 2\nu^3 t \cdot \boldsymbol{\mu} \otimes \boldsymbol{\mu} \otimes \boldsymbol{\mu}^T.$$

As we have seen above either the result on the compound model presented in Theorem 1 or the result on the Brownian motion subordination presented in Theorem 3 could be used to derive the third cumulant for the considered aggregated claim model. In the next section we discuss the model in which both the results are needed.

3.4 The Poisson-normal inverse Gaussian claim model

We start from the consideration of the multivariate Brownian motion $\mathbf{B}_d$ with drift $\boldsymbol{\mu} \in \mathrm{R}^d$ and covariance matrix $\boldsymbol{\Sigma} \in \mathrm{R}^{d\times d}$ that is subordinated to the random variable Γ that has generalized inverse Gaussian (GIG) distribution with parameters $\lambda \in \mathrm{R}$, $\chi > 0$, and $\psi > 0$, i.e. $\Gamma \sim GIG(\lambda, \chi, \psi)$. We work under the assumption that $\mathbf{B}_d$ and Γ are independently distributed. Hence, $\mathbf{B}_d(\Gamma)$ has the d-dimensional generalized hyperbolic distribution with parameters λ, χ, ψ, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$. This distribution is denoted by $GH_d(\boldsymbol{\Sigma}, \boldsymbol{\mu}, \lambda, \chi, \psi)$. Then the third cumulant for $\mathbf{B}_d(\Gamma)$ can be obtained from Theorem 3 and from the formula for the moments of $\Gamma \sim GIG(\lambda, \chi, \psi)$ (see Lemma 1 of Scott et al. (2011)) which are given by

$$E\left(\Gamma^k\right) = \left(\frac{\chi}{\psi}\right)^{(\lambda+k)/2} \frac{K_{\lambda+k}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})},$$

where $K_a(\cdot)$ is the modified Bessel function of the third kind (see Abramowitz and Stegun (1972)).

Corollary 5. *Let $\mathbf{B}_d$ be a d-variate Brownian motion with drift $\boldsymbol{\mu} \in \mathrm{R}^d$ and covariance matrix $\boldsymbol{\Sigma} \in \mathrm{R}^{d\times d}$ that is subordinated to the random variable $\Gamma \sim GIG(\lambda, \chi, \psi)$. Then the third cumulant of $\mathbf{B}_d(\Gamma)$ is given by*

$$\begin{aligned}\kappa_3(\mathbf{B}_d(\Gamma)) \quad = \quad & \frac{1}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\chi}{\psi}\right)^{(\lambda+2)/2} \Big[K_{\lambda+2}(\sqrt{\chi\psi}) \cdot (\boldsymbol{\Sigma}^T\boldsymbol{\mu}^T + \boldsymbol{\mu}\otimes\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\otimes\boldsymbol{\mu}) \\ & + \sqrt{\frac{\chi}{\psi}} K_{\lambda+3}(\sqrt{\chi\psi}) \cdot \boldsymbol{\mu}\otimes\boldsymbol{\mu}\otimes\boldsymbol{\mu}^T\Big].\end{aligned}$$

When $\lambda = -1/2$, Γ has the inverse Gaussian (IG) distribution with parameters χ and ψ, i.e. $\Gamma \sim IG(\chi, \psi)$. Thus, $\mathbf{B}_d(\Gamma)$ has the d-dimensional normal inverse Gaussian (NIG) distribution that is the special case of GH distribution, and its third cumulant can be easily derived from Corollary 5 and the well-known properties of the modified Bessel function (see Kotz et al. (2001) for example)

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}, \quad K_{-\lambda}(x) = K_\lambda(x), \quad K_{c+1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{k=0}^{c}\frac{(c+k)!}{(c-k)!k!}(2x)^{-k},$$

where c is non-negative integer.

Corollary 6. *Let $\mathbf{B}_d$ be a d-variate Brownian motion with drift $\boldsymbol{\mu} \in \mathrm{R}^d$ and covariance matrix $\boldsymbol{\Sigma} \in \mathrm{R}^{d\times d}$ that is subordinated to the random variable $\Gamma \sim IG(\chi, \psi)$. Then the* *third cumulant of* $\mathbf{B}_d(\Gamma)$ *is given by*

$$\begin{aligned}\kappa_3(\mathbf{B}_d(\Gamma)) &= \left(\frac{\chi}{\psi}\right)^{3/4}\left[\left(1+\frac{1}{\sqrt{\chi\psi}}\right)\cdot(\boldsymbol{\Sigma}^T\boldsymbol{\mu}^T+\boldsymbol{\mu}\otimes\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\otimes\boldsymbol{\mu})\right.\\ &\quad\left.+\sqrt{\frac{\chi}{\psi}}\left(1+\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\right)\cdot\boldsymbol{\mu}\otimes\boldsymbol{\mu}\otimes\boldsymbol{\mu}^T\right].\end{aligned}$$

The important model of multivariate aggregate claims can be obtained when the number of claim sizes have the Poisson-inverse Gaussian distribution, and the claim sizes have the multivariate NIG distribution, i.e.

$$\mathbf{S}(t)=\mathbf{B}_d(\Gamma(N(t)))=\sum_{k=1}^{N(t)}(\mathbf{B}_d(\Gamma(k))-\mathbf{B}_d(\Gamma(k-1)))=\sum_{k=1}^{N(t)}\mathbf{X}_k, \tag{13}$$

where $\mathbf{X}_k$'s are identically and indpendently distributed claims that have a multivariate NIG distribution with parameters $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}$, χ, and ψ.

Next, we apply the above results to derive the third cumulant of $\mathbf{S}(t)$ in (13).

Corollary 7. *For the model (13) we have the following formula for the third cumulant*

$$\begin{aligned}\kappa_3(\mathbf{S}(t)) &= t\left(\frac{\chi}{\psi}\right)^{3/4}\left[\left(1+\frac{1}{\sqrt{\chi\psi}}\right)\cdot(\boldsymbol{\Sigma}^T\boldsymbol{\mu}^T+\boldsymbol{\mu}\otimes\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\otimes\boldsymbol{\mu})\right.\\ &\quad\left.+2\sqrt{\frac{\chi}{\psi}}\left(1+\frac{3}{\sqrt{\chi\psi}}+\frac{3}{\chi\psi}\right)\cdot\boldsymbol{\mu}\otimes\boldsymbol{\mu}\otimes\boldsymbol{\mu}^T\right].\end{aligned}$$

4 Summary

In this paper we analyzed the model for the aggregated multivariate claims when the aggregating variable is independent of the multivariate claims. We derived the third cumulant for the general case involving random compounding and random subordination. These general results are used to obtain specific formula for concrete models. In the first one, the multivariate aggregate claims with multivariate claim sizes which have the SN distribution and the count which has a Poisson distribution are considered. The second case focuses on the GAL distribution and when the aggregation is made by a negative binomial variable. Moreover, we established relation between an asymmetric Laplace motion and a negative binomial process that corresponds to the invariance principle of the aggregating claims for the GAL distribution. Finally, we provided continuous time versions of the results.

5 Appendix

Proof of Theorem 1. From (7) we have that $\mathbf{S} = \mathbf{X}_1 + ... + \mathbf{X}_N$ and its mean is equal to $\nu_1 \boldsymbol{\xi}_1$. Then the third cumulant of $\mathbf{S}$ is given by

$$\kappa_3(\mathbf{S}) = E\left[(\mathbf{S} - \nu_1 \boldsymbol{\xi}_1) \otimes (\mathbf{S} - \nu_1 \boldsymbol{\xi}_1)^T \otimes (\mathbf{S} - \nu_1 \boldsymbol{\xi}_1)\right],$$

which may be represented as $E\left[(\mathbf{w} + \mathbf{u}) \otimes (\mathbf{w} + \mathbf{u})^T \otimes (\mathbf{w} + \mathbf{u})\right]$ with $\mathbf{w} = (\mathbf{X}_1 - \boldsymbol{\xi}_1) + \cdots + (\mathbf{X}_N - \boldsymbol{\xi}_1)$ and $\mathbf{u} = \boldsymbol{\xi}_1 (N - \nu_1)$.

Using the standard properties of the Kronecker product and the vectorization operator we obtain that

$$\begin{aligned} \kappa_3(\mathbf{S}) &= E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{w}\right) + E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{u}\right) + E\left(\mathbf{w}\mathbf{u}^T \otimes \mathbf{w}\right) + E\left(\mathbf{w}\mathbf{u}^T \otimes \mathbf{u}\right) \\ &\quad + E\left(\mathbf{u}\mathbf{w}^T \otimes \mathbf{w}\right) + E\left(\mathbf{u}\mathbf{w}^T \otimes \mathbf{u}\right) + E\left(\mathbf{u}\mathbf{u}^T \otimes \mathbf{w}\right) + E\left(\mathbf{u}\mathbf{u}^T \otimes \mathbf{u}\right). \end{aligned}$$

First, we evaluate $E\left(\mathbf{w} \otimes \mathbf{w}^T \otimes \mathbf{w}\right)$. Since $\{\mathbf{X}_i\}_{i=1}^N$ and N are independently distributed we get that $E\left(\mathbf{w} \otimes \mathbf{w}^T \otimes \mathbf{w} \middle| N = n\right)$ is the third cumulant of the sum $\mathbf{X}_1 + ... + \mathbf{X}_n$, that is

$$E\left\{\left[\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1)\right] \otimes \left[\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1)\right]^T \otimes \left[\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1)\right]\right\}.$$

Because $\mathbf{X}_1, ..., \mathbf{X}_n$ are independent and identically distributed, then the third cumulant of their sum equals to the third cumulant of the $i-$th summand $\mathbf{X}_i$, multiplied by the number of summands n, i.e. $E\left(\mathbf{w} \otimes \mathbf{w}^T \otimes \mathbf{w} \middle| N = n\right) = n \cdot \boldsymbol{\xi}_3$. By taking expectations over N we obtain that $E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{w}\right) = \nu_1 \cdot \boldsymbol{\xi}_3$.

Next, we evaluate $E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{u}\right)$ starting from the conditional expectation

$$E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{u} \middle| N = n\right) = E\left\{\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1) \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1)^T \otimes [\boldsymbol{\xi}_1 (n - \nu_1)]\right\}.$$

Applying the properties of linear operators we get

$$E\left(\mathbf{w}\mathbf{w}^T \otimes \mathbf{u} \middle| N = n\right) = (n - \nu_1) E\left[\sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1) \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\xi}_1)^T\right] \otimes \boldsymbol{\xi}_1.$$

The above expectation is the second cumulant of the sum $\mathbf{X}_1 + ... + \mathbf{X}_n$. The random vectors $\mathbf{X}_1 ..., \mathbf{X}_n$ are independent and identically distributed, so that the second cumulant of their sum equals the second cumulant of the $i-$th summand $\mathbf{X}_i$, multiplied by the number of summands n:

$$E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{u}\middle|\, N=n\right)=(n-\nu_1)\cdot n\cdot\boldsymbol{\xi}_2\otimes\boldsymbol{\xi}_1.$$

By taking expectations over N we obtain

$$E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{u}\right)=E\left\{E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{u}\middle|\, N\right)\right\}=E\left[(N-\nu_1)\cdot N\right]\cdot\boldsymbol{\xi}_2\otimes\boldsymbol{\xi}_1.$$

The expectation in the right-hand side of the above equation is just the variance (that is the second cumulant) of N. Hence $E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{u}\right)=\nu_2\cdot\boldsymbol{\xi}_2\otimes\boldsymbol{\xi}_1$.

We shall now evaluate $E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{w}\right)$, starting from the identity

$$E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{w}\middle|\, N=n\right)=E\left\{\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\otimes\left[\boldsymbol{\xi}_1(n-\nu_1)\right]^T\otimes\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\right\},$$

which may be simplified as follows, by remembering that $\mathbf{a}\otimes\mathbf{b}^T=\mathbf{b}^T\otimes\mathbf{a}$, for any two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{w}\middle|\, N=n\right)=E\left\{\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\otimes\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\otimes\left[\boldsymbol{\xi}_1(n-\nu_1)\right]^T\right\}.$$

Properties of linear operators lead to

$$E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{w}\middle|\, N=n\right)=E\left[\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\otimes\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\right]\otimes\left[\boldsymbol{\xi}_1(n-\nu_1)\right]^T.$$

For any vector $\mathbf{a}$, $\mathbf{a}\otimes\mathbf{a}$ is just the matrix $\mathbf{a}\mathbf{a}^T$, vectorized. Hence

$$E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{w}\middle|\, N=n\right)=E\left\{\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)\otimes\sum_{i=1}^{n}(\mathbf{X}_i-\boldsymbol{\xi}_1)^T\right\}^V\otimes\left[\boldsymbol{\xi}_1(n-\nu_1)\right]^T.$$

The above expectation is the second cumulant of the sum $\mathbf{X}_1+\cdots+\mathbf{X}_n$. An argument similar to the one used before leads to

$$E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{w}\right)=E\left\{E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{w}\middle|\, N\right)\right\}=E\left[N\boldsymbol{\xi}_2^V\otimes\boldsymbol{\xi}_1^T(N-\nu_1)\right]=\nu_2\cdot\boldsymbol{\xi}_2^V\otimes\boldsymbol{\xi}_1^T.$$

In order to evaluate $E\left(\mathbf{u}\otimes\mathbf{w}^T\otimes\mathbf{w}\right)$, we shall apply linear properties of expectation to obtain $E\left(\mathbf{u}\mathbf{w}^T\otimes\mathbf{w}\middle|\, N=n\right)=uE\left(\mathbf{w}^T\otimes\mathbf{w}\middle|\, N=n\right)$. Recall now that $\mathbf{w}^T\otimes\mathbf{w}=\mathbf{w}\mathbf{w}^T=\mathbf{w}\otimes\mathbf{w}^T$ and use arguments similar to the above ones to obtain $E\left(\mathbf{u}\otimes\mathbf{w}^T\otimes\mathbf{w}\right)=\nu_2\cdot\boldsymbol{\xi}_1\otimes\boldsymbol{\xi}_2$.

We shall now consider the expectation $E\left(\mathbf{u}\otimes\mathbf{u}^T\otimes\mathbf{u}\right)$, that is

$$E\left\{\left[\boldsymbol{\xi}_1\left(n-\nu_1\right)\right]\otimes\left[\boldsymbol{\xi}_1\left(n-\nu_1\right)\right]^T\otimes\left[\boldsymbol{\xi}_1\left(n-\nu_1\right)\right]\right\}=E\left[\left(N-\nu_1\right)^3\right]\boldsymbol{\xi}_1\otimes\boldsymbol{\xi}_1^T\otimes\boldsymbol{\xi}_1.$$

The expectation in the right-hand side of the above equation is the third cumulant of N. Hence $E\left(\mathbf{u}\otimes\mathbf{u}^T\otimes\mathbf{u}\right)=\nu_3\boldsymbol{\xi}_1\otimes\boldsymbol{\xi}_1^T\otimes\boldsymbol{\xi}_1$.

We shall now consider the expectation $E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{u}\right)$. The identity

$$E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{u}\middle|\,N=n\right)=E\left\{\left[\sum_{i=1}^{n}\left(\mathbf{X}_i-\boldsymbol{\xi}_1\right)\right]\otimes\left[\boldsymbol{\xi}_1\left(n-\nu_1\right)\right]^T\otimes\boldsymbol{\xi}_1\left(n-\nu_1\right)\right\}$$

may be simplified by linear properties of the expectation into

$$E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{u}\middle|\,N=n\right)=\left\{\sum_{i=1}^{n}\left[E\left(\mathbf{X}_i\right)-\boldsymbol{\xi}_1\right]\right\}\otimes\left[\boldsymbol{\xi}_1\left(n-\nu_1\right)\right]^T\otimes\boldsymbol{\xi}_1\left(n-\nu_1\right).$$

By definition, $E\left(\mathbf{X}_i\right)=\boldsymbol{\xi}_1$, so that $E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{u}\middle|\,N=n\right)=\mathbf{O}$, where $\mathbf{O}$ is a $d^2\times d$ matrix of zeros. As a direct consequence, $E\left(\mathbf{w}\otimes\mathbf{u}^T\otimes\mathbf{u}\right)=\mathbf{O}$. In a similar way, we can prove that $E\left(\mathbf{u}\mathbf{w}^T\otimes\mathbf{u}\right)=E\left(\mathbf{u}\mathbf{u}^T\otimes\mathbf{w}\right)=\mathbf{O}$.

We shall now complete the proof by representing the third cumulant of $\mathbf{S}$ as the sum of $E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{w}\right)$, $E\left(\mathbf{w}\mathbf{w}^T\otimes\mathbf{u}\right)$, $E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{w}\right)$, $E\left(\mathbf{w}\mathbf{u}^T\otimes\mathbf{u}\right)$, $E\left(\mathbf{u}\mathbf{w}^T\otimes\mathbf{w}\right)$, $E\left(\mathbf{u}\mathbf{w}^T\otimes\mathbf{u}\right)$, $E\left(\mathbf{u}\mathbf{u}^T\otimes\mathbf{w}\right)$, $E\left(\mathbf{u}\mathbf{u}^T\otimes\mathbf{u}\right)$:

$$\kappa_3\left(\mathbf{S}\right)=\nu_1\boldsymbol{\xi}_3+\nu_2\left(\boldsymbol{\xi}_2\otimes\boldsymbol{\xi}_1+\boldsymbol{\xi}_2^V\boldsymbol{\xi}_1^T+\boldsymbol{\xi}_1\otimes\boldsymbol{\xi}_2\right)+\nu_3\boldsymbol{\xi}_1\otimes\boldsymbol{\xi}_1^T\otimes\boldsymbol{\xi}_1.$$

□

Proof of Theorem 2. We shall first prove part (a) of the Theorem. The first three cumulants of $\mathbf{X}_i\sim SN_d\left(\boldsymbol{\Omega},\boldsymbol{\alpha}\right)$ are

$$\boldsymbol{\xi}_1=\sqrt{\frac{2}{\pi}}\boldsymbol{\delta},\quad \boldsymbol{\xi}_2=\boldsymbol{\Omega}-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T,\quad \boldsymbol{\xi}_3=\sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}.$$

By assumption, N is a Poisson random variate, whose first three cumulants equal the parameter λ: $\nu_1=\nu_2=\nu_3=\lambda$. We shall now apply Theorem 1 to obtain the third cumulant of the first N components of the sequence:

$$\kappa_3\left(\mathbf{S}\right)=\lambda\sqrt{\frac{2}{\pi}}\left[\left(\frac{4}{\pi}-1\right)\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}+\left(\boldsymbol{\Omega}-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T\right)\otimes\boldsymbol{\delta}\right.$$
$$\left.+\left(\boldsymbol{\Omega}-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T\right)^V\boldsymbol{\delta}^T+\boldsymbol{\delta}\otimes\left(\boldsymbol{\Omega}-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T\right)+\frac{2}{\pi}\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}\right].$$

The identities $\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}=\boldsymbol{\delta}\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}=\left(\boldsymbol{\delta}\boldsymbol{\delta}^T\right)^V\boldsymbol{\delta}^T=\boldsymbol{\delta}\otimes\boldsymbol{\delta}\boldsymbol{\delta}^T$ and a little algebra help in simplifying the right-hand side of the above equation as follows:

$$\kappa_3(\mathbf{S}) = \lambda\sqrt{\frac{2}{\pi}}\left(\mathbf{\Omega}\otimes\boldsymbol{\delta}+\mathbf{\Omega}^V\boldsymbol{\delta}^T+\boldsymbol{\delta}\otimes\mathbf{\Omega}-\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}\right).$$

We shall now prove part (b) of the Theorem. The variance of $\mathbf{S}$ is

$$V(\mathbf{S}) = \nu_1\boldsymbol{\xi}_2+\nu_2\boldsymbol{\xi}_1\boldsymbol{\xi}_1^T = \lambda\left(\mathbf{\Omega}-\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T\right)+\lambda\frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}^T = \lambda\mathbf{\Omega}.$$

Hence the third standardized cumulant of $\mathbf{S}$, denoted by $\kappa_3\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]$, is

$$\sqrt{\frac{2}{\lambda\pi}}\left(\mathbf{\Omega}^{-1/2}\otimes\mathbf{\Omega}^{-1/2}\right)\left(\mathbf{\Omega}\otimes\boldsymbol{\delta}+\mathbf{\Omega}^V\boldsymbol{\delta}^T+\boldsymbol{\delta}\otimes\mathbf{\Omega}-\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta}\right)\mathbf{\Omega}^{-1/2}.$$

The product $\kappa_3\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]\sqrt{2/(\lambda\pi)}$ might be expressed as

$$\begin{gathered}\mathbf{\Omega}^{-1/2}\mathbf{\Omega}\mathbf{\Omega}^{-1/2}\otimes\mathbf{\Omega}^{-1/2}\boldsymbol{\delta}+\left(\mathbf{\Omega}^{-1/2}\otimes\mathbf{\Omega}^{-1/2}\right)\mathbf{\Omega}^V\boldsymbol{\delta}^T\mathbf{\Omega}^{-1/2}+\\ \mathbf{\Omega}^{-1/2}\boldsymbol{\delta}\otimes\mathbf{\Omega}^{-1/2}\mathbf{\Omega}\mathbf{\Omega}^{-1/2}-\mathbf{\Omega}^{-1/2}\boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\mathbf{\Omega}^{-1/2}\otimes\mathbf{\Omega}^{-1/2}\boldsymbol{\delta},\end{gathered}$$

by remembering that $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$, $\mathbf{\Omega}=1\otimes\mathbf{\Omega}=\mathbf{\Omega}\otimes 1$ and $\mathbf{a}\otimes\mathbf{a}^T\otimes\mathbf{a}=\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}^T$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are matrices of appropriate order and $\mathbf{a}$ is a vector. By definition, $\boldsymbol{\eta}=\mathbf{\Omega}^{-1/2}\boldsymbol{\delta}$ and $\mathbf{\Omega}^{-1/2}\mathbf{\Omega}\mathbf{\Omega}^{-1/2}=\mathbf{I}$. Moreover, $\left(\mathbf{\Omega}^{-1/2}\otimes\mathbf{\Omega}^{-1/2}\right)\mathbf{\Omega}^V=\left(\mathbf{\Omega}^{-1/2}\mathbf{\Omega}\mathbf{\Omega}^{-1/2}\right)^V$ by elementary properties of the Kronecker product and vectorization. Hence the third standardized cumulant of $\mathbf{S}$ is

$$\kappa_3\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]=\sqrt{\frac{2}{\lambda\pi}}\left(\mathbf{I}\otimes\boldsymbol{\eta}+\mathbf{I}^V\boldsymbol{\eta}^T+\boldsymbol{\eta}\otimes\mathbf{I}-\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right).$$

We shall now prove the third part of the Theorem. Mardia's skewness is the squared Frobenius norm of the third standardized cumulant:

$$\beta_{1,d}^M(\mathbf{S})=\left\|\kappa_3\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]\right\|^2=tr\left\{\kappa_3^T\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]\kappa_3\left[(\lambda\mathbf{\Omega})^{-1/2}\mathbf{S}\right]\right\}.$$

By part (b) of the Theorem, $(\lambda\pi/2)\,\beta_{1,d}^M(\mathbf{S})$ is the trace of

$$\left(\mathbf{I}\otimes\boldsymbol{\eta}+\mathbf{I}^V\boldsymbol{\eta}^T+\boldsymbol{\eta}\otimes\mathbf{I}-\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)\left(\mathbf{I}\otimes\boldsymbol{\eta}^T+\boldsymbol{\eta}\mathbf{I}^{VT}+\boldsymbol{\eta}^T\otimes\mathbf{I}-\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right).$$

We shall first evaluate $T_1 = tr\left[(\mathbf{I}\otimes\boldsymbol{\eta})\left(\mathbf{I}\otimes\boldsymbol{\eta}^T+\boldsymbol{\eta}\mathbf{I}^{VT}+\boldsymbol{\eta}^T\otimes\mathbf{I}-\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]$:

$$
\begin{gathered}
tr\left[(\mathbf{I}\otimes\boldsymbol{\eta})\left(\mathbf{I}\otimes\boldsymbol{\eta}^T\right)\right]+tr\left[(\mathbf{I}\otimes\boldsymbol{\eta})\,\boldsymbol{\eta}\mathbf{I}^{VT}\right]+tr\left[(\mathbf{I}\otimes\boldsymbol{\eta})\left(\boldsymbol{\eta}^T\otimes\mathbf{I}\right)\right]\\
-tr\left[(\mathbf{I}\otimes\boldsymbol{\eta})\left(\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]=tr\left(\mathbf{I}\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right)+tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\right)\\
+tr\left(\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)-tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right)=d\cdot\boldsymbol{\eta}^T\boldsymbol{\eta}+\boldsymbol{\eta}^T\boldsymbol{\eta}+\boldsymbol{\eta}^T\boldsymbol{\eta}-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2.
\end{gathered}
$$

We shall now evaluate $T_2 = tr\left[\mathbf{I}^V\boldsymbol{\eta}^T\left(\mathbf{I}\otimes\boldsymbol{\eta}^T+\boldsymbol{\eta}\mathbf{I}^{VT}+\boldsymbol{\eta}^T\otimes\mathbf{I}-\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]$:

$$
\begin{gathered}
tr\left[\left(\mathbf{I}^V\boldsymbol{\eta}^T\right)\left(\mathbf{I}\otimes\boldsymbol{\eta}^T\right)\right]+tr\left[\left(\mathbf{I}^V\boldsymbol{\eta}^T\right)\left(\boldsymbol{\eta}\mathbf{I}^{VT}\right)\right]\\
+tr\left[\left(\mathbf{I}^V\boldsymbol{\eta}^T\right)\left(\boldsymbol{\eta}^T\otimes\mathbf{I}\right)\right]-tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right)\\
=tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\right)+\boldsymbol{\eta}^T\boldsymbol{\eta}\cdot\mathbf{I}^{VT}\mathbf{I}^V+tr\left(\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2\\
=\boldsymbol{\eta}^T\boldsymbol{\eta}+\boldsymbol{\eta}^T\boldsymbol{\eta}\cdot d+\boldsymbol{\eta}^T\boldsymbol{\eta}-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2.
\end{gathered}
$$

We shall now evaluate $T_3 = tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\mathbf{I}\otimes\boldsymbol{\eta}^T+\boldsymbol{\eta}\mathbf{I}^{VT}+\boldsymbol{\eta}^T\otimes\mathbf{I}-\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]$:

$$
\begin{aligned}
& tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\mathbf{I}\otimes\boldsymbol{\eta}^T\right)\right]+tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\boldsymbol{\eta}\mathbf{I}^{VT}\right)\right]+\\
& tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\boldsymbol{\eta}^T\otimes\mathbf{I}\right)\right]-tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]\\
= & tr\left(\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)+tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\right)+tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\mathbf{I}\right)-tr\left(\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right)\\
= & \boldsymbol{\eta}^T\boldsymbol{\eta}+\boldsymbol{\eta}^T\boldsymbol{\eta}+d\cdot\boldsymbol{\eta}^T\boldsymbol{\eta}-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2.
\end{aligned}
$$

We shall now evaluate $T_4 = tr\left[(\boldsymbol{\eta}\otimes\mathbf{I})\left(\mathbf{I}\otimes\boldsymbol{\eta}^T+\boldsymbol{\eta}\mathbf{I}^{VT}+\boldsymbol{\eta}^T\otimes\mathbf{I}-\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]$:

$$
\begin{gathered}
tr\left[\left(\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)\left(\mathbf{I}\otimes\boldsymbol{\eta}^T\right)\right]+tr\left[\left(\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)\left(\boldsymbol{\eta}\mathbf{I}^{VT}\right)\right]+\\
tr\left[\left(\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)\left(\boldsymbol{\eta}^T\otimes\mathbf{I}\right)\right]-tr\left[\left(\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\right)\left(\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\otimes\boldsymbol{\eta}^T\right)\right]\\
=tr\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right]+\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)tr\left[(\boldsymbol{\eta}\otimes\boldsymbol{\eta})\left(\mathbf{I}^{VT}\right)\right]+tr\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right]\\
-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)tr\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\otimes\boldsymbol{\eta}\boldsymbol{\eta}^T\right]=\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2+\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2+\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2-\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^3.
\end{gathered}
$$

All the above traces are functions of $\boldsymbol{\eta}^T\boldsymbol{\eta}$, which equals q. In fact, by definition,

$$
\boldsymbol{\delta}=\frac{\boldsymbol{\Omega}\boldsymbol{\alpha}}{\sqrt{1+\boldsymbol{\alpha}^T\boldsymbol{\Omega}\boldsymbol{\alpha}}},\ \boldsymbol{\eta}=\boldsymbol{\Omega}^{-1/2}\boldsymbol{\delta}\text{ and }q=\frac{\boldsymbol{\alpha}^T\boldsymbol{\Omega}\boldsymbol{\alpha}}{1+\boldsymbol{\alpha}^T\boldsymbol{\Omega}\boldsymbol{\alpha}}.
$$

Mardia's skewness is just the sum of T_1, T_2, T_3, and T_4, multiplied by $2/(\pi\lambda)$. A little algebra leads to

$$
\beta_{1,d}^M(\mathbf{S})=\frac{2q}{\lambda\pi}\left[3(d-1)+(q-3)^2\right].
$$

We shall now focus on partial skewness. Loperfido (2015a) showed that it is the squared norm of

$$\left(\mathbf{I} \otimes \boldsymbol{\eta}^T + \boldsymbol{\eta}\mathbf{I}^{VT} + \boldsymbol{\eta}^T \otimes \mathbf{I} - \boldsymbol{\eta}^T \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta}^T\right)\mathbf{I}^V =$$
$$\left(\mathbf{I} \otimes \boldsymbol{\eta}^T\right)\mathbf{I}^V + \boldsymbol{\eta}\mathbf{I}^{VT}\mathbf{I}^V + \left(\boldsymbol{\eta}^T \otimes \mathbf{I}\right)\mathbf{I}^V - \left(\boldsymbol{\eta}^T \otimes \boldsymbol{\eta}\boldsymbol{\eta}^T\right)\mathbf{I}^V.$$

The identity

$$\left(\mathbf{I} \otimes \boldsymbol{\eta}^T + \boldsymbol{\eta}\mathbf{I}^{VT} + \boldsymbol{\eta}^T \otimes \mathbf{I} - \boldsymbol{\eta}^T \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta}^T\right)\mathbf{I}^V = \boldsymbol{\eta}^{TV} + d\boldsymbol{\eta} + \boldsymbol{\eta}^V + \left(\boldsymbol{\eta}\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^V$$

follows from a fundamental property of the Kronecker product and the vectorization operator: $(\mathbf{ABC})^V = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\mathbf{B}^V$, when $\mathbf{A} \in \mathbb{R}^p \times \mathbb{R}^q$, $\mathbf{B} \in \mathbb{R}^q \times \mathbb{R}^r$, $\mathbf{C} \in \mathbb{R}^r \times \mathbb{R}^s$. We also have

$$\left(\mathbf{I} \otimes \boldsymbol{\eta}^T + \boldsymbol{\eta}\mathbf{I}^{VT} + \boldsymbol{\eta}^T \otimes \mathbf{I} - \boldsymbol{\eta}^T \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta}^T\right)\mathbf{I}^V = \left(d + 2 + \boldsymbol{\eta}^T\boldsymbol{\eta}\right)\boldsymbol{\eta},$$

since $\mathbf{ab}^T$, $\mathbf{a} \otimes \mathbf{b}^T$ and $\mathbf{b}^T \otimes \mathbf{a}$ denote the same matrix, when $\mathbf{a}$ and $\mathbf{b}$ are two vectors. By definition, $q = \boldsymbol{\eta}^T\boldsymbol{\eta}$, so that the partial skewness $\beta_{1,d}^P(\mathbf{S})$ is

$$\left\|\sqrt{\frac{2}{\lambda\pi}}\left(\mathbf{I} \otimes \boldsymbol{\eta}^T + \boldsymbol{\eta}\mathbf{I}^{VT} + \boldsymbol{\eta}^T \otimes \mathbf{I} - \boldsymbol{\eta}^T \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta}^T\right)\mathbf{I}^V\right\|^2 = \frac{2q}{\lambda\pi}(d + 2 + q)^2.$$

We shall now focus on directional skewness, by considering the cubic form γ, defined as

$$\left(\mathbf{v}^T \otimes \mathbf{v}^T\right)\left(\mathbf{I} \otimes \boldsymbol{\eta} + \mathbf{I}^V\boldsymbol{\eta}^T + \boldsymbol{\eta} \otimes \mathbf{I} - \boldsymbol{\eta} \otimes \boldsymbol{\eta}^T \otimes \boldsymbol{\eta}\right)\mathbf{v} = \left(\mathbf{v}^T\boldsymbol{\eta}\right)\left[3\left(\mathbf{v}^T\mathbf{v}\right) - \left(\mathbf{v}^T\boldsymbol{\eta}\right)^2\right].$$

The last identity follows from repeated application of the previously recalled properties of the Kronecker product and the vectorization operator. By differentiating γ with respect to $\mathbf{v}^T\boldsymbol{\eta}$ we obtain

$$\frac{\partial\gamma}{\partial\left(\mathbf{v}^T\boldsymbol{\eta}\right)} = \frac{3\left[\left(\mathbf{v}^T\mathbf{v}\right) - \left(\mathbf{v}^T\boldsymbol{\eta}\right)^2\right]}{\left(\mathbf{v}^T\boldsymbol{\eta}\right)^2\left[3\left(\mathbf{v}^T\mathbf{v}\right) - \left(\mathbf{v}^T\boldsymbol{\eta}\right)^2\right]^2}.$$

The numerator of the fraction is positive, due to the inequality $\left(\mathbf{v}^T\boldsymbol{\eta}\right)^2 < \left(\mathbf{v}^T\mathbf{v}\right)$, which in turn follows from the Cauchy-Schwarz inequality $\left(\mathbf{v}^T\boldsymbol{\eta}\right)^2 \leq \left(\mathbf{v}^T\mathbf{v}\right)\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)$ and the squared norm $\boldsymbol{\eta}^T\boldsymbol{\eta} = q = \boldsymbol{\alpha}^T\boldsymbol{\Omega}\boldsymbol{\alpha}/\left(1 + \boldsymbol{\alpha}^T\boldsymbol{\Omega}\boldsymbol{\alpha}\right)$ being smaller than one. As a direct consequence, γ is an increasing function of $\mathbf{v}^T\boldsymbol{\eta}$, which attains its maximum value when $\mathbf{v}$ is proportional to $\boldsymbol{\eta}$.

Let $\mathbf{Z} = (\lambda\boldsymbol{\Omega})^{-1/2}[\mathbf{S} - E(\mathbf{S})]$ be the standardized version of $\mathbf{S}$. The Pearson's skewness of the linear combination $\mathbf{v}^T\mathbf{Z}$ is

$$\beta_1\left(\mathbf{v}^T\mathbf{Z}\right) = \frac{2}{\lambda\pi}\frac{\left[\left(\mathbf{v}^T \otimes \mathbf{v}^T\right)\kappa_3(\mathbf{Z})\mathbf{v}\right]^2}{\left(\mathbf{v}^T\mathbf{v}\right)^3} = \frac{2}{\lambda\pi}\frac{\gamma^2}{\left(\mathbf{v}^T\mathbf{v}\right)^3}.$$

It attains its maximum value when $\mathbf{v}$ is proportional to $\boldsymbol{\eta}$:

$$\beta_1\left(\boldsymbol{\eta}^T\mathbf{Z}\right) = \frac{2}{\lambda\pi}\frac{\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2\left[3\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right) - \left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^2\right]^2}{\left(\boldsymbol{\eta}^T\boldsymbol{\eta}\right)^3} = \frac{2q\left(3-q\right)^2}{\lambda\pi}.$$

Since Pearson's skewness is invariant with respect to affine transformations, we have $\beta_1\left(\boldsymbol{\eta}^T\mathbf{Z}\right) = \beta_{1,d}^D\left(\mathbf{S}\right)$ and this completes the proof. □

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220 07 Lund, Sweden