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SECOND-ORDER CONTINUOUS TIME MOVING AVERAGES VIA SPECTRAL REPRESENTATION

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Abstract. The spectral representation of a moving average process obtained as a convolution of a kernel with a general noise measure is studied. A proof of the spectral theorem that yields explicit expression for the spectral measure in terms of the noise measure is presented. The main interest is in noise measures generated by second order Lévy motions. For practical considerations, such measures are easily available through independent sampling. On the other hand spectral measures are not since their increments are dependent, with the notable exception of the Gaussian noise case. For this reason the issue of approximating the spectral measure by independent increments of the noise is also addressed. For the purpose of approximating the moving average process through sums of trigonometric functions, the mean square error of discretization of the spectral representation is assessed. For a specified accuracy, the coefficients of approximation are explicitly given. The method is illustrated for moving averages processes driven by Laplace motion.

Keywords: generalized Laplace distribution; moving average processes; weakly stationary second order processes; spectral representation

AMS Subject Classification: 60G10, 60G12

1. Introduction

Despite being one of the most fundamental results in the theory of weakly stationary processes, the spectral representation has not been explored outside the class of Gaussian processes. In fact, we are not aware of any explicit presentation of the spectral measure for non-Gaussian processes. However in engineering sciences, mostly due to the efficiency of the Fast Fourier Transform technique, spectral theory has played a central role in studying random signals. The interest in the method, driven mainly by stochastic problems in signal processing, engineering mechanics and structural engineering led to a large body of work where transformed Gaussian models were utilized to construct new classes of weakly stationary processes. We refer to [5] and [6] and references therein for recent developments in the area. In particular, in [6] a construction based on the spectral representation that consists of a superposition of harmonics with uncorrelated but dependent random amplitudes is proposed. In [5], memoryless transformations of Gaussian processes, and transformations with memory of the Brownian and Lévy processes were introduced. The difficulty with this method, as pointed in [7], is that the spectral density and the marginal distribution are linked together through the underlying transformation, i.e., the spectral density of the Gaussian process is not the same as the spectral density of the transformed process. A different class of models constructed in [7], using the spectral representation but with the sinusoidal frequencies being modeled as random variables, disentangled the spectral density from the marginal distribution. However, the actual spectral representation of this model is not known as of now.

An alternative approach to constructing non-Gaussian processes is based on stochastic integration with respect to Lévy motions. For example, a new type of processes that are expressed as moving averages with respect to Lévy measure driven by Laplace noise, and shorty referred to as Laplace moving average (LMA) processes are introduced and studied in [1] and [2]. Our main goal in this paper is to provide with explicit forms of both the spectral measure and the spectral representation for these LMA processes.

In the first part of the paper we focus on general properties that hold for any moving average process with respect to orthogonal stochastic measures on the real line. After we establish notation we give a direct proof of the spectral representation for a LMA process and with an explicit form of its spectral measure in terms of the kernel functions. Next, we approximate the stochastic spectral measure by means of the stochastic orthogonal measure that appears in the definition of the LMA process. This allows effective approximation of vectors of independent increments of the spectral measure and therefore mean square approximation of the original LMA process through its spectral representation. Finally, the paper concludes by remarking how the Laplace stochastic measure can be obtained either by independent increment property or by pure jump series representation of Laplace motion. All these results combined together provide an effective approach to approximate stochastic processes using decomposition to harmonics as in the spectral representation theorem.

2. Spectral representation of moving average processes

One of the most popular classes of weakly stationary processes are the moving averages defined as

$$X(t) = \int f(t+s)\ d\Lambda(s). \tag{1}$$

The kernel f is in $L^2(\mathbb{R})$ – the Hilbert space of complex-valued square integrable functions on the real line with inner product $\int f(s)\overline{g(s)}\ ds$ and corresponding norm $\|f\|_2$. Further Λ is an orthogonal stochastic measure on the Borel sets on the real line $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, that is controlled by the Lebesgue measure λ so that, for each Borel set $A \in \mathcal{B}(\mathbb{R})$, $\Lambda(A)$ is in the Hilbert space of complex-valued random variables with finite second moments on some probability space, and $Var(\Lambda(A)) = \lambda(A)$. Recall that the mapping $g \in L^2(\mathbb{R}) \mapsto \int g\ d\Lambda$ is an isometry between $L^2(\mathbb{R})$ and the closure in the mean square sense of the linear space spanned of random variables $\Lambda(A)$, $A \in \mathcal{B}(\mathbb{R})$, that from now on is denoted by $L^2(\Lambda)$.

Each weakly stationary process has a spectral representation, i.e. a representation of the form

$$X(t) = \int e^{i\omega t}\ dZ(\omega) \tag{2}$$

where Z is a stochastic spectral measure. While one can obtain directly this spectral representation from Bochner's theorem, here for the sake of introducing notation, we sketch the argument for the spectral representation through the Fourier transform of the kernel in a moving average process. In certain cases, working with the Fourier transform instead of directly with the problem is advantageous. See for example [3], where the authors studied the crossings of the shot noise process with jumps by working on the Fourier transform of the function that maps each level to its mean number of crossings.

We initiate this presentation by introducing some rather standard notation. For any $f \in L^2(\mathbb{R})$, define the Fourier transform by

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int e^{-is\omega} f(s)\ ds, \tag{3}$$

while $\check{f}$ is the inverse Fourier transform so that $\check{\widehat{f}} = f$ and

$$\check{f}(s) = \int e^{i\omega s} f(\omega) d\omega.$$

For any Borel set A let $\mathbb{I}_A$ denote its indicator function. Finally we define a continuous operator $\check{\mathbb{I}}_A *$ on $L^2(\mathbb{R})$ valid for any function $\phi \in L^2(\mathbb{R})$ through

$$\check{\mathbb{I}}_A * \check{\phi} = \widetilde{\mathbb{I}_A \cdot \phi}. \tag{4}$$

The most fundamental properties of the so defined family of operators are presented in Appendix A. We turn now to the spectral representation of the so-called Laplace moving average process.

For a fixed kernel $f \in L^2(\mathbb{R})$, we introduce now a random measure Z_f that will be later identified as the spectral measure for the moving average process (1). For a Borel set A, we define

$$Z_f(A) = \int \check{\mathbb{I}}_A * f(s)\ d\Lambda(s). \tag{5}$$

Notice that the integral in (5) is well defined since the operator $\check{\mathbb{I}}_A * f$ is in $L^2(\mathbb{R})$, see Lemma 1 in Appendix A.

Theorem 1. *The function $A \to Z_f(A)$ given by (5) defines a σ-additive orthogonal stochastic measure on Borel sets. This measure is controlled by $\nu_f(A) = 2\pi \int_A |\widehat{f}(\omega)|^2\ d\omega$. Moreover, for each $f \in L^2(\mathbb{R})$ we have*

$$Var\left(\int e^{i\omega t}\ dZ_f(\omega)\right) = \|f\|_2^2,$$

i.e. for each $t \in \mathbb{R}$ the mapping $f \mapsto \int e^{i\omega t}\ dZ_f(\omega)$ is an isometry from $L^2(\mathbb{R})$ to $L^2(\Lambda)$.

Proof. Since the mapping $g \in L^2(\mathbb{R}) \mapsto \int g\ d\Lambda \in L^2(\Lambda)$ is an isometry, the orthogonality and σ-additivity of Z_f both are direct consequences of Lemma 1 of Appendix A. We also have

$$Var\left(Z_f(A)\right) = \|\check{\mathbb{I}}_A * f\|_2^2 = 2\pi \|\mathbb{I}_A \widehat{f}\|_2^2,$$

while the isometric identity follows from

$$Var\left(\int e^{i\omega t}\ dZ_f(\omega)\right) = \nu_f(\mathbb{R}) = 2\pi \|\widehat{f}\|_2^2 = \|f\|_2^2.$$

□

Let us note the following fundamental fact:

Proposition 1. *For each $f \in L^2(\mathbb{R})$ we have the equality of the following two $L^2(\Lambda)$ elements*

$$\int f(s)\ d\Lambda(s) = Z_f(\mathbb{R}).$$

Proof. By (4), for a function f in $L^2(\mathbb{R})$ we have

$$\check{\mathbb{I}}_{\mathbb{R}} * f(s) = \widetilde{\mathbb{I}_{\mathbb{R}} \cdot \widehat{f(s)}} = \check{\widehat{f}}(s) = f(s) = \int_{\mathbb{R}} e^{i\omega s} \widehat{f}(\omega)\ d\omega$$

so that

$$\int f(s)\ d\Lambda(s) = \int \check{\mathbb{I}}_{\mathbb{R}} * f(s)\ d\Lambda(s) = Z_f(\mathbb{R}),$$

where the last equality follows from (5). □

We are now ready to formulate the spectral theorem for the moving average process. We note that the presented approach to the spectral theorem is a consequence of the properties of the Fourier transform and does not rely on less explicit argument following from the Bochner theorem (which is not used here). This approach through explicitelly defined stochastic spectral measure Z_f allows us to write approximations to the moving average process as discussed in Section 3.

Theorem 2. *Let f be a function in $L^2(\mathbb{R})$ and Z_f be defined in (5). The moving average process given in (1) can be represented as*

$$X(t) = \int e^{it\omega}\ dZ_f(\omega),$$

where for each $t \in \mathbb{R}$ the equality is of elements of $L^2(\Lambda)$.

Proof. To prove the theorem we shall establish an isomorphism between three Hilbert spaces. The first one, $\mathcal{H}_X$, is the space spanned by linear combinations of the random variables $X(t)$, $t \in \mathbb{R}$ and all their limits in the mean square sense. In $\mathcal{H}_X$, we consider the usual inner product $(Y_1, Y_2) = E(Y_1 \cdot \overline{Y}_2)$. The second space, $\mathcal{H}_{\nu_f}$, is defined for a fixed $f \in L^2(\mathbb{R})$ as the space that is spanned by the non-random complex-valued functions g such that

$$\int |g(\omega)|^2\ d\nu_f(\omega) < \infty,$$

where ν_f is defined in Theorem 1. This forms a Hilbert space when equipped with the inner product

$$(g_1, g_2)_{\nu_f} = \int g_1(\omega)\overline{g_2(\omega)}\ d\nu_f(\omega) = 2\pi \int g_1(\omega)\overline{g_2(\omega)}|\widehat{f}(\omega)|^2\ d\omega. \tag{6}$$

The third space, $\mathcal{H}_f$ (for the same fixed f) contains all elements in $L^2(\mathbb{R})$ of the form $\breve{e}_t * f(s) = \int e^{i\omega(t+s)}\widehat{f}(\omega)\ d\omega$, their finite linear combinations and limits in the $L^2(\mathbb{R})$-norm. The inner product is the standard one in $L^2(\mathbb{R})$, see Appendix A. We note also that

$$\breve{e}_t * f(s) = f(t+s),$$

since $f(t+s) = \check{\widehat{f}}(t+s)$.

The isomorphism between $\mathcal{H}_X$ and $\mathcal{H}_f$ maps $X(t)$ to the element $\breve{e}_t * f(s) = f(t+s)$ which in turn is mapped to $x_t(\omega) = e^{i\omega t}$, to establish an isomorphism between $\mathcal{H}_f$ and $\mathcal{H}_{\nu_f}$. Note that from the properties of the Fourier transform $x_t \in \mathcal{H}_{\nu_f}$ since $\int |e^{i\omega t}|^2\ d\nu_f = 2\pi \int |\widehat{f}(\omega)|^2\ d\omega < \infty$.

The corresponding inner products are preserved as shown

$$\begin{aligned}
(X(t), X(\tau)) &= E\left(X(t) \cdot \overline{X(\tau)}\right) = f * \tilde{f}(t-\tau) = \widehat{\widehat{f * \tilde{f}}}(t-\tau) = \int e^{i\omega(t-\tau)}\widehat{f * \tilde{f}}(\omega)\ d\omega = \\
&= 2\pi \int e^{i\omega(t-\tau)}\widehat{f}(\omega)\widehat{f}(-\omega)\ d\omega = 2\pi \int e^{i\omega(t-\tau)}\widehat{f}(\omega)\overline{\widehat{f}(\omega)}\ d\omega = \\
&= (x_t, x_\tau)_{\nu_f},
\end{aligned}$$

where $\tilde{f}(u) = \overline{f(-u)}$, and since f is real $\widehat{f}(-\omega) = \overline{\widehat{f}(\omega)}$.

On the other hand,

$$(\breve{e}_t * f, \breve{e}_\tau * f) = \int f(t+s)\overline{f(\tau+s)}ds = f * \tilde{f}(t-\tau) = (X(t), X(\tau)).$$

These correspondences, which are one to one, can be extended to both finite linear combinations and limits with respect to the respective norms, which shows that the spaces $\mathcal{H}_X$, $\mathcal{H}_{\nu_f}$ and $\mathcal{H}_f$ are isomorphic.

For a measurable set $A \subset \mathbb{R}$, the operator:

$$\breve{\mathbb{I}}_A * f(s) = \int_A e^{i\omega s}\widehat{f}(\omega)\ d\omega \tag{7}$$

is an element in $\mathcal{H}_f$, while

$$Z_f(A) = \int \breve{\mathbb{I}}_A * f(s)\ d\Lambda(s) \tag{8}$$

is an element in $\mathcal{H}_X$. The following argument is standard but for completeness in presentation we sketch it. By Stone-Weierstrass Theorem (plus some standard arguments), there is a sequence $h_{k,T}(\omega)$ of linear combinations of $x_{t_i}(\omega)$ that is uniformly bounded for all (k,T) by a constant $M \geqslant 1$ for which for any fixed $T > 0$:

$$\lim_{k\to\infty} \int_{[-T,T]} |\mathbb{I}_A(\omega) - h_{k,T}(\omega)|^2 \, |\widehat{f}(\omega)|^2 d\omega = 0. \tag{9}$$

From that we can find k_T that converges to infinity as T increases without bound such that

$$\int |\mathbb{I}_A(\omega) - h_{k_T,T}(\omega)|^2 \, |\widehat{f}(\omega)|^2 d\omega \leqslant 2M \int_{[-T,T]^c} |\widehat{f}(\omega)|^2 d\omega + \int_{[-T,T]} |\mathbb{I}_A(\omega) - h_{k_T,T}(\omega)|^2 \, |\widehat{f}(\omega)|^2 d\omega$$

and the first term in the right hand side converges to zero with T increasing without bound because $\widehat{f}(\omega)$ is square integrable while the second term converges to zero because of (9) and a proper choice of k_T. This shows that $\mathbb{I}_A(\omega)$ is an element in $\mathcal{H}_{\nu_f}$ and by the isometries the corresponding elements in $\mathcal{H}_f$ and $\mathcal{H}_X$ must be the ones given in (7) and (8) respectively.

Thus if a simple function $h = \sum a_i \mathbb{I}_{A_i}$ approximates x_t in $\mathcal{H}_{\nu_f}$, then, due to the isometry, $\int h \, dZ_f$ approximates both $\int x_t \, dZ_f$ and $X(t)$, implying that these last two must be equal in $\mathcal{H}_X$, and the result of the theorem follows immediately by the choice $x_t = e^{it}$. □

3. Approximation of the spectral measure

A method to obtain the values of the spectral measure Z_f becomes a fundamental issue when one intends to approximate moving average processes. While independently scattered measure Λ is typically available through independent sampling from the corresponding infinitely divisible distribution, $Z_f(A)$'s are uncorrelated but not necessarily independent and the issue of sampling is therefore non-trivial. In this section we discuss how to approximate $Z_f(A)$ by some linear function of $\Lambda(B_1), \ldots, \Lambda(B_m)$ for appropriately chosen disjoint sets $B_1, \ldots, B_m$. In this and the following sections, we assume that the independently scattered measures have finite second moments, like it is the case with the generalized Laplace measures related to Laplace motion, see [1] and Section 5.

We start by remarking that since the spectral measures are expressed as integrals with respect to an independently scattered measure, it is in principle possible to obtain such approximation by discretizing the integrals. However, this would not provide with an obvious explicit expression for the coefficients of approximation in terms of the kernel. In this section given the kernel and accuracy of approximation we obtain explicit form of coefficients. The method is based on the two lemmas that are placed in Appendix B together with the proof of the following result.

Proposition 2. *For $\epsilon > 0$, and kernel f there exists a grid $\mathcal{S} = (s_k)_{k=-M}^{M+1}$ such that for each interval $A \subset \mathbb{R}$ there is a sequence of complex numbers $(\gamma_k)_{k=-M}^{M}$ such that*

$$\left(E \left| Z_f(A) - \sum_{k=-M}^{M} \gamma_k \Lambda(s_k, s_{k+1}] \right|^2 \right)^{1/2} < \epsilon.$$

Remark 1. It is of interest, for example for computational purposes, to extract from the proofs of Lemmas 2, 3 and Proposition 2 an explicit form of γ_k. In fact we have

$$\gamma_k = \sum_{l=1}^{J} \widehat{f}(\tilde{u}_l) \frac{e^{iu_l^R s_k} - e^{iu_l^L s_k}}{is_k}, \; k = -M, \ldots, M,$$

where the intervals $(u_l^L, u_l^R]$, s_k and $\tilde{u}_l$, are obtained as follows. Let $\mathcal{W} = (\omega_j)_{j=-N}^{N+1}$ be a grid such that

$$\sum_{j=-N}^{N} \int_{\omega_j}^{\omega_{j+1}} |\widehat{f}(\omega) - \widehat{f}(\tilde{\omega}_j)|^2 d\omega + \int_{[\omega_N, \omega_{N+1}]^c} |\widehat{f}(\omega)|^2 d\omega < \epsilon^2,$$

where $\tilde{\omega}_j = (\omega_j + \omega_{j+1})/2$. Let also δ be the minimum step of grid $\mathcal{W}$. Additionally let us select $\delta_1 > 0$ and $K > 0$ such that for every finite partition $\mathcal{U} = (u_k)_{k=-M}^{M+1}$ of $[-K, K]$ with diameter less than δ_1:

$$\int_{-\infty}^{\infty} \left(\frac{\sin u}{u} - \sum_{k=-M}^{M} \frac{\sin u_k}{u_k} \mathbb{I}_{(u_k, u_{k+1}]}(u) \right)^2 dx < \epsilon^2. \tag{10}$$

Further let $\delta_2 > 0$ be chosen so that for each $z \in \mathbb{C}$ with $|z| < \delta_2$:

$$|1 - e^z| < \frac{\epsilon}{4\sqrt{\delta\pi}}. \tag{11}$$

Finally, with this choice of K, δ, δ_1, δ_2, ω_{-N}, ω_{N+1}, the grid $\mathcal{S} = (s_k)_{k=-M}^{M+1}$ is chosen over $[-K/\delta, K/\delta]$ with the diameter smaller than $\min(\delta_1/\delta, \delta_2/\max(|\omega_{-N}|, |\omega_{N+1}|))$. We note that the choice of $\mathcal{S}$ is independent of A.

Then $(u_l^L, u_l^R]$, dependent on A, are non-empty intervals among $A \cap (\omega_j, \omega_{j+1}]$, i.e. if there is J of them, and enumerate them as $(u_l^L, u_l^R]$ and take $\tilde{u}_l = \tilde{\omega}_j$ for $l = 1, \ldots, J$, see Appendix B for further details.

4. Spectral approximation of moving average process

A general moving average process can be approximated in two essentially different ways. The first one is based on its formulation as a moving average given in (1) and involves discretized convolution of the kernel with the increments of the random measure

$$X(t) \approx X_{\mathcal{S}}(t) \stackrel{def}{=} \sum_{i=-M}^{M} f(t + s_i)\, d\Lambda_i, \tag{12}$$

where $d\Lambda_i = \Lambda((s_i, s_{i+1}])$. The second one can use the spectral representation given in (2) and approximate the process through a sum of trigonometric functions with random coefficients

$$X(t) \approx X_{\mathcal{W}}(t) \stackrel{def}{=} \sum_{j=-N}^{N} e^{i\tilde{\omega}_j t} dZ_j^f, \tag{13}$$

where $dZ_j^f = Z_f((\omega_j, \omega_{j+1}])$ and $\tilde{\omega}_j = (\omega_j + \omega_{j+1})/2$. In general, the second method represents weakly stationary approximation of the process and thus is non-trivial over its entire domain, while the first one limits applicability of the representation only to a certain compact domain. This can be seen by evaluating the mean square errors.

The first approximation has mean square error given by

$$E\,|X(t) - X_{\mathcal{S}}(t)|^2 = \int_{(s_{-M}, s_{M+1}]^c} |f(t+s)|^2\, ds + \sum_{i=-M}^{M} \int_{s_i}^{s_{i+1}} |f(t+s) - f(t+s_i)|^2\, ds,$$

where the second term on the right hand side can be made arbitrarily small by choosing a fine grid. However, the first term will be large whenever t is large enough in absolute value to move the main mass of the mapping $s \mapsto |f(t+s)|^2$ beyond the interval $(s_{-M}, s_M]$.

The mean square error for the second approximation is given by

(14)

$$E\left|X(t)-X_{\mathcal{W}}(t)\right|^2 = 2\pi\int_{(\omega_{-N},\omega_{N+1}]^c}\left|\widehat{f}(\omega)\right|^2\,d\omega + 2\pi\sum_{j=-N}^{N}\int_{\omega_j}^{\omega_{j+1}}\left|e^{i\omega t}-e^{i\tilde{\omega}_j t}\right|^2\left|\widehat{f}(\omega)\right|^2\,d\omega.$$

The first term on the right hand side is independent of t and can be made small by taking sufficiently large interval $[\omega_{-N},\omega_N]$. Assuming a kernel f such that its Fourier transform is continuous on the real line, then for $M_j = \max_{\omega\in(\omega_j,\omega_{j+1}]}\left|\widehat{f}(\omega)\right|^2$:

$$\begin{aligned}
\sum_{j=-N}^{N}\int_{\omega_j}^{\omega_{j+1}}\left|e^{i\omega t}-e^{i\tilde{\omega}_j t}\right|^2\left|\widehat{f}(\omega)\right|^2\,d\omega &\leqslant \sum_{j=-N}^{N}\max_{\omega\in(\omega_j,\omega_{j+1}]}\left|\widehat{f}(\omega)\right|^2\int_{\omega_j}^{\omega_{j+1}}\left|1-e^{i(\tilde{\omega}_j-\omega)t}\right|^2\,d\omega\\
&= 2\sum_{j=-N}^{N}M_j\left(\omega_{j+1}-\omega_j-\int_{\omega_j}^{\omega_{j+1}}\cos((\tilde{\omega}_j-\omega)t)\,d\omega\right)\\
&= 2\sum_{j=-N}^{N}M_j(\omega_{j+1}-\omega_j)\left(1-\frac{\sin((\omega_{j+1}-\omega_j)t/2)}{(\omega_{j+1}-\omega_j)t/2}\right)\\
&\leqslant 2\left(\int_{-\omega_{-N}}^{\omega_{N+1}}\left|\widehat{f}\right|^2(u)\,du+\epsilon\right)M(\delta,t),
\end{aligned}$$

where ϵ is the absolute error of approximation of $\int_{\omega_{-N}}^{\omega_{N+1}}\left|\hat{f}\right|^2(u)\,du$ by $\sum_{j=-N}^{N}M_j(\omega_{j+1}-\omega_j)$ and

$$\delta = \max_{j=-N,\ldots,N}(\omega_{j+1}-\omega_j)/2,\quad M(\delta,t)=\sup_{u\leqslant\delta}\left(1-\frac{\sin(ut)}{ut}\right). \tag{15}$$

We can summarize this in the following result.

Theorem 3. *Let f be in $L^2(\mathbb{R})$ and $\mathcal{W}=(\omega_j)_{j=-N}^{N+1}$ be a finite grid. Denote $\widehat{f}_{\mathcal{W}} = \widehat{f}\,\mathbb{I}_{[\omega_{-N},\omega_{N+1})}$ and $M_j = \max_{\omega\in(\omega_j,\omega_{j+1}]}\left|\widehat{f}(\omega)\right|^2$. Let $\delta>0$ and $M(\delta,t)$ be given as in (15). Then for each $t\in\mathbb{R}$:*

$$E\left|X(t)-X_{\mathcal{W}}(t)\right|^2 \leqslant 2\pi\left(\left\|\widehat{f}-\widehat{f}_{\mathcal{W}}\right\|^2+2M(\delta,t)\left(\left\|\widehat{f}_{\mathcal{W}}\right\|^2+\epsilon\right)\right),$$

where ϵ is the absolute value of difference between $\left\|\widehat{f}_{\mathcal{W}}\right\|^2$ and $\sum_{j=-N}^{N}M_j(\omega_{j+1}-\omega_j)$.

Although we have shown that for both approximations the mean square error can be made arbitrarily small only over argument values in a compact set, there is a fundamental difference between the two approximations. The first one is not stationary and the approximating process eventually dies out when the argument increases in absolute value – the approximation holds within a compact domain but fails when the entire range of the argument is considered. However, while the second approximation is also valid only within a compact set if one considers the mean square error as a criterion of the closeness between the two processes, the process is additionally stationary. Thus, while the mean square error is not small over the entire real domain, due to the weak stationarity of the original process and its approximation, they are close to each other in the distributional sense over the entire domain. We conclude that the second approximation is advantageous because it is not limited in representing random behavior of the approximated process only to an interval.

Remark 2. Using the above bound, the two processes $X(t)$ and $X_{\mathcal{W}}(t)$ can be made arbitrarily and uniformly close in the mean square sense over $[-T,T]$, for a certain T, by first chosing the end points ω_{-N} and ω_{N+1} of the grid so that $\|\widehat{f}-\widehat{f}_N\|^2$ is small. By continuity of $1-\sin x/x$ at zero, for a fixed T there is $\delta>0$ such that $M(\delta,t)$ is small for all $t\in[-T,T]$. It is also clear that δ can be selected so small that ϵ is small for each grid with the diameter smaller than 2δ.

5. Sampling from random Laplace measure

All the results presented in the previous sections were independent of the choice of the second order stochastic measure with respect to which the moving average process was defined. However, our main interest is in the non-Gaussian models called Laplace moving averages that are obtained by filtering an asymmetric Laplace noise $dL(t)$ through a kernel. The asymmetric Laplace motion $L(t)$ is best described as a gamma mean and variance model, i.e.

$$L(t)=\mu\Gamma(t)+B(\Gamma(t)), \tag{16}$$

where $\Gamma(t)$ has a gamma distribution with shape t/ν and scale 1, and $B(t)$ is Brownian motion with parameter σ. These processes have been extensively discussed in [1] and [8]. However, for reader's convenience and completeness of presentation, we briefly discuss methods of generating the random vector $\mathbf{\Lambda}=(\Lambda(s_k,s_{k+1}])_{k=-M}^{M}$. This is equivalent to providing a method of generating the values of the Laplace motion since $\Lambda(a,b]=L(b)-L(a)$ for an interval $(a,b]\subset[0,\infty)$. This can be done in two different ways.

5.1. **Approximation by increments.** Using 16 is probably the most straightforward way to approximate the Laplace motion as summarized in the following algorithm.

- Pick a grid $\mathcal{S}=(s_k)_{k=-M}^{M+1}$.
- Approximate $2M+1$ i.i.d. $\Gamma(s_k/\nu,1)$ random variables and store them in a vector G.
- Approximate $2M+1$ i.i.d. zero mean Gaussian random variables having variances $\sigma^2G_j, j=1,\dots,2M+1$ and store them in a vector B.
- Compute $X=\mu\cdot G+B$.

5.2. **Approximation by series expansion.** An alternative method of approximating Laplace motion is to use series representations. Here we give a short account of such an approach. Recall, the shot-noise series expansion of a Gamma process as given in [4]

$$\Gamma(t)=\sum_{j=1}^{\infty}e^{-\nu\gamma_j/T}W_j\mathbb{I}_{(0,t]}(U_j),\quad 0\leqslant t\leqslant T$$

where $\{W_j\}$ is a sequence of i.i.d standard exponential random variables independent of $\{\gamma_j\}$,the arrival times of a Poisson process with intensity 1, and of $\{U_j\}$ a sequence of independent uniformly distributed random variables in $(0,T]$.

By using this representation in (16) we obtain the following series expansion of the asymmetric Laplace motion

$$L(t)=\mu\sum_{j=1}^{\infty}e^{-\nu\gamma_j/T}W_j\mathbb{I}_{(0,t]}(U_j)+\sum_{j=1}^{\infty}Z_j\sqrt{e^{-\nu\gamma_j/T}W_j}\mathbb{I}_{(0,t]}(U_j),\quad 0\leqslant t\leqslant T. \tag{17}$$

Taking a finite sum instead of the infinite series results in removing the small jumps of the Lévy process. Therefore a simple simulation algorithm is given by:

- Pick a grid $\mathcal{S}=(s_k)_{k=-M}^{M+1}$ and J the number of terms in the series (17).
- Approximate $(2M+1)\times J$ i.i.d. zero mean Gaussian random variables having variances σ^2 and store them in a matrix B.

- Approximate $(2M+1)\times J$ i.i.d. standard exponential random variables and store them in a matrix W.
- Approximate $(2M+1)\times J$ values from a Poisson process with intensity 1 and store them in a matrix G.
- Approximate $(2M+1)\times J$ i.i.d. uniform on $(s_{-M}, s_{M+1}]$ random variables and create a matrix $U=[u_{ij}]$ where for every j the entry $u_{i\cdot}$ is 1 if $s_{i-1} < u_{i\cdot} \leqslant s_i$ and 0 otherwise.
- Compute $X = B.*\sqrt{exp(-\nu * G/T).*W}.*U$, where $.*$ denotes coordinate multiplication, and store them in a matrix X.
- Sum the elements of matrix X along each row to obtain an approximation of $L(s_k), k=-M,\dots,M+1$.

Appendix A. Vector valued measures through Fourier transform

For a relatively compact set A on $\mathbb{R}$ and any function $\phi\in L^2(\mathbb{R})$ the following relation is true

$$\widetilde{\mathbb{I}_A\cdot\phi} = \check{\mathbb{I}}_A * \check{\phi}.$$

This relation can be in a certain sense extended to an arbitrary Borel set A. Namely, there is an operator on $L^2(\mathbb{R})$ that can be denoted as $\check{\mathbb{I}}_A*$ corresponding to the bounded operator $\phi\mapsto \mathbb{I}_A\phi$. This operator can be written in the following more explicit way

$$\check{\mathbb{I}}_A * f(s) = \widetilde{\mathbb{I}_A\cdot\widehat{f}}(s) = \int_A e^{i\omega s}\widehat{f}(\omega)\ d\omega. \tag{18}$$

The following facts about the above defined operator are almost immediate.

For a real-valued function f, we have $\widehat{f}(-\omega)=\widehat{f}(\omega)$, and thus

$$\check{\mathbb{I}}_{-A} * f(-s) = \int_{-A} e^{i\omega(-s)}\widehat{f}(\omega)\ d\omega = \int_A e^{i\omega s}\widehat{f}(\omega)\ d\omega = \check{\mathbb{I}}_A * f(s). \tag{19}$$

For a complex-valued function f we have

$$\check{\mathbb{I}}_A * f(s) = \int_A e^{i\omega s}\widehat{f}(\omega)\ d\omega = \int_{-A} \overline{e^{i\omega s}}\,\overline{\widehat{f}(\omega)}\ d\omega = \overline{\check{\mathbb{I}}_{-A} * f(s)}, \tag{20}$$

where the overline denotes the conjugate of a complex number.

The following lemma is a standard fact from the spectral theory for operators on Hilbert spaces. For completeness, we include its proof.

Lemma 1. *The function of a Borel set with values in the algebra of continuous operators on $L^2(\mathbb{R})$ that is given by Eq. 18 defines an additive orthogonal projector valued measure that is pointwise σ-additive on the entire line, i.e. for each $f\in L^2(\mathbb{R})$ the vector valued set function $A\mapsto \check{\mathbb{I}}_A * f$ is σ-additive.*

Proof. We start with showing that for each Borel A the operator $\check{\mathbb{I}}_A*$ is a continuous (bounded) orthonormal projection, i.e. that

$$\left(\check{\mathbb{I}}_A*\right)^2 = \check{\mathbb{I}}_A*,$$
$$\|\check{\mathbb{I}}_A * \| \leqslant 1.$$

The first condition follows from the following

$$\begin{aligned}\check{\mathbb{I}}_A * \check{\mathbb{I}}_A * f &= \int_A e^{i\omega s}\,\widehat{\mathbb{I}\check{}_A * f}(\omega)\ d\omega\\ &= \int_A e^{i\omega s}\cdot\widehat{\widehat{\mathbb{I}_{\mathbb{A}}\cdot\widehat{f}}}(\omega)\ d\omega\\ &= \check{\mathbb{I}}_A * f.\end{aligned}$$

We remind that $L^2(\mathbb{R})$ is equipped with the usual inner product defined as

$$(f,g) = \int f(s)\overline{g(s)}ds$$

for any f and g in $L^2(\mathbb{R})$, and with the norm given by

$$||f||_2^2 = (f,f)$$

for any f in $L^2(\mathbb{R})$.

Then the second one is a direct consequence of the fact that the Fourier transform is an isometry on $L^2(\mathbb{R})$:

$$\|\check{\mathbb{I}}_A * f\|_2 = \sqrt{2\pi}\|\mathbb{I}_A \cdot \widehat{f}\|_2 \leqslant \sqrt{2\pi}\|\widehat{f}\|_2 = \|f\|_2.$$

Let A, B be two disjoint Borel sets on the real line. Then $\check{\mathbb{I}}_A$ and $\check{\mathbb{I}}_B$ are projecting on orthogonal subspaces of $L^2(\mathbb{R})$. Indeed, for each $f \in L^2(\mathbb{R})$:

$$\int \check{\mathbb{I}}_A * f(s) \cdot \overline{\check{\mathbb{I}}_B * f(s)}\ ds = 2\pi \int \mathbb{I}_A(\omega)\widehat{f}(\omega) \cdot \mathbb{I}_B(\omega)\overline{\widehat{f}(\omega)}\ d\omega = 0, \tag{21}$$

where the equality of the integrals is again a result of the isometric property.

Since the additivity of $A \mapsto \check{\mathbb{I}}_A *$ follows from the additivity of the Fourier transform and its inverse, in order to show the σ - additivity, it is enough to show the continuity condition. More specifically, for an increasing sequence $\{A_n\}$ of Borel subsets with $A = \bigcup_{n=1}^{\infty} A_n$, we need to show convergence of $\check{\mathbb{I}}_{A_n} * f$ to $\check{\mathbb{I}}_A * f$ as n goes to infinity. This follows from

$$\|\check{\mathbb{I}}_A * f - \check{\mathbb{I}}_{A_n} * f\|_2 = \sqrt{2\pi}\|\mathbb{I}_{A\setminus A_n} \cdot \widehat{f}\|_2 \leqslant \|f\|_2 \tag{22}$$

and the convergence of $\|\mathbb{I}_{A\setminus A_n} \cdot \widehat{f}\|_2$ is assured by the dominated convergence theorem, since f (and thus also $\widehat{f}$) is a square integrable function. □

Appendix B. Technical lemmas and proofs

Lemma 2. *Let $\epsilon > 0$, and $\mathcal{W} = (\omega_j)_{j=-N}^{N+1}$ be a grid such that for*

$$\widehat{f}_{\mathcal{W}}(\omega) = \sum_{j=-N}^{N} \widehat{f}(\tilde{\omega}_j)\mathbb{I}_{(\omega_j,\omega_{j+1}]}(\omega),$$

with $\tilde{\omega}_j = (\omega_j + \omega_{j+1})/2$, we have

$$\sqrt{2\pi}\left\|\widehat{f} - \widehat{f}_{\mathcal{W}}\right\|_2 < \epsilon. \tag{23}$$

Then for each measurable A:

$$\left(E\left|Z_f(A) - \sum_{j=-N}^{N} \widehat{f}(\tilde{\omega}_j)\int \check{\mathbb{I}}_{(\omega_j,\omega_{j+1}]\cap A}(s)\ d\Lambda(s)\right|^2\right)^{1/2} < \epsilon.$$

Proof. Let us fix a measurable set A, then

$$\left(E\left|Z_f(A)-\sum_{j=-N}^{N}\widehat{f}(\tilde{\omega}_j)\int\check{\mathbb{I}}_{(\omega_j,\omega_{j+1}]\cap A}(s)\ d\Lambda(s)\right|^2\right)^{1/2}=\left\|\widehat{\mathbb{I}}_A*f-\sum_{j=-N}^{N}\widehat{f}(\tilde{\omega}_j)\check{\mathbb{I}}_{(\omega_j,\omega_{j+1}]\cap A}\right\|_2$$
$$=\left(2\pi\int_A\left|\widehat{f}(\omega)-\sum_{j=-N}^{N}\widehat{f}(\tilde{\omega}_j)\mathbb{I}_{(\omega_j,\omega_{j+1}]}(\omega)\right|^2\ d\omega\right)^{1/2}\leqslant\sqrt{2\pi}\|\widehat{f}-\widehat{f}_{\mathcal{W}}\|<\epsilon,$$

where the first equality is a result of the isomorphism between $L^2(\Lambda)$ and $L^2(\mathbb{R})$, the second equality is the isomorphism between functions and their Fourier transforms, and the third relation results by replacing A by $\mathbb{R}$. □

Lemma 3. *For each $\epsilon>0$, $\delta>0$, and real endpoints ω_L, ω_R there exists a finite grid $\mathcal{S}=(s_k)_{k=-M}^{M+1}$ such that*

$$\sup_{\substack{\omega\in[\omega_L,\omega_R]\\ \tilde{\delta}\in(0,\delta]}}\left(E\left|\int\check{\mathbb{I}}_{(\omega-\tilde{\delta},\omega+\tilde{\delta}]}(s)\ d\Lambda(s)-\sum_{k=-M}^{M}\frac{e^{i(\omega+\tilde{\delta})s_k}-e^{i(\omega-\tilde{\delta})s_k}}{is_k}\Lambda(s_k,s_{k+1}]\right|^2\right)^{1/2}<\epsilon.$$

Proof. Let us fix $\epsilon>0$ and $\delta>0$. From the square integrability of $\sin u/u$, there is $\delta_1>0$ and $K>0$ such that for every finite partition, say $\mathcal{U}=(u_k)_{k=-M}^{M}$ of $[-K,K]$ with diameter less than δ_1:

$$\int_{-\infty}^{\infty}\left(\frac{\sin u}{u}-\sum_{k=-M}^{M}\frac{\sin u_k}{u_k}\mathbb{I}_{(u_k,u_{k+1}]}(u)\right)^2\ du<\epsilon^2/16. \tag{24}$$

From now on we assume that δ_1 and K are chosen so that the above holds.

By the continuity of e^z and since $\int\sin^2 x/x^2\ dx=\pi$, there exists $\delta_2>0$ such that for each $z\in\mathbb{C}$ with $|z|<\delta_2$:

$$|1-e^z|<\frac{\epsilon}{4\sqrt{\delta\pi}}. \tag{25}$$

Let us choose a grid $\mathcal{S}=(s_k)_{k=-M}^{M}$ of the interval $(-K/\delta,K/\delta)$ with diameter smaller than $\min(\delta_1/\delta,\delta_2/\max(|\omega_R|,|\omega_L|))$, fix an arbitrary $\omega\in[\omega_L,\omega_R]$ and take $\tilde{\delta}<\delta$. Then notice that

$$\int\check{\mathbb{I}}_{(\omega-\tilde{\delta},\omega+\tilde{\delta}]}(s)\ d\Lambda(s)=\int\int_{\omega-\tilde{\delta}}^{\omega+\tilde{\delta}}e^{i\nu s}\ d\nu\ d\Lambda(s)=2\int e^{i\omega s}\frac{\sin(\tilde{\delta}s)}{s}\ d\Lambda(s).$$

Consequently, by the isometry and then the triangle inequality

$$\frac{1}{2}\left(E\left|\int \check{\mathbb{I}}_{(\omega-\tilde{\delta},\omega+\tilde{\delta}]}(s)\ d\Lambda(s) - \sum_{k=-M}^{M} \frac{e^{i(\omega+\tilde{\delta})s_k} - e^{i(\omega-\tilde{\delta})s_k}}{is_k}\left(\Lambda(s_{k+1}) - \Lambda(s_k)\right)\right|^2\right)^{1/2} =$$

$$= \left(E\left|\int e^{i\omega s}\frac{\sin(\tilde{\delta}s)}{s}\ d\Lambda(s) - \sum_{k=-M}^{M} e^{i\omega s_k}\frac{\sin(\tilde{\delta}s_k)}{s_k}\left(\Lambda(s_{k+1}) - \Lambda(s_k)\right)\right|^2\right)^{1/2} =$$

$$= \left(E\left|\int e^{i\omega s}\frac{\sin(\tilde{\delta}s)}{s}\ d\Lambda(s) - \sum_{k=-M}^{M} e^{i\omega s_k}\frac{\sin(\tilde{\delta}s_k)}{s_k}\left(\Lambda(s_{k+1}) - \Lambda(s_k)\right)\right|^2\right)^{1/2} =$$

$$= \left(\int\left|e^{i\omega s}\frac{\sin(\tilde{\delta}s)}{s} - \sum_{k=-M}^{M} e^{i\omega s_k}\frac{\sin(\tilde{\delta}s_k)}{s_k}\mathbb{I}_{[s_k,s_{k+1})}(s)\right|^2 ds\right)^{1/2}$$

$$\leqslant \left(\int\left|\sum_{k=-M}^{M}\left(e^{i\omega s} - e^{i\omega s_k}\right)\mathbb{I}_{[s_k,s_{k+1})}(s)\frac{\sin(\tilde{\delta}s)}{s}\right|^2 ds\right)^{1/2} +$$

$$+ \left(\int\left|\frac{\sin(\tilde{\delta}s)}{s} - \sum_{k=-M}^{M}\frac{\sin(\tilde{\delta}s_k)}{s_k}\mathbb{I}_{[s_k,s_{k+1})}(s)\right|^2 ds\right)^{1/2}$$

$$= \left(\int\sum_{k=-M}^{M}\left|1 - e^{i\omega(s_k-s)}\right|^2\mathbb{I}_{[s_k,s_{k+1})}(s)\frac{\sin^2(\tilde{\delta}s)}{s^2}ds\right)^{1/2} +$$

$$+ \left(\int\left|\frac{\sin(u)}{u} - \sum_{k=-M}^{M}\frac{\sin(u_k)}{u_k}\mathbb{I}_{[u_k,u_{k+1})}(u)\right|^2 du\right)^{1/2},$$

where $u_k = \tilde{\delta}s_k$. By (25) and since $|\omega(s_k - s)| \leqslant \max(|\omega_R|, |\omega_L|)|s_k - s| \leqslant \delta_2$, and $\int \frac{\sin^2(\tilde{\delta}s)}{s^2}\ ds = \tilde{\delta}\pi$ the term in the second last line of the above can be bounded by $\epsilon/4$. Further, since $|u_k - u_{k+1}| = \tilde{\delta}|s_k - s_{k+1}| \leqslant \delta|s_k - s_{k+1}| < \delta_1$, the last term is smaller than $\epsilon/4$ by (24), which concludes the proof. □

Proof of Proposition 2. Let $\mathcal{W}$ be chosen as in Lemma 2 with $\epsilon/2$. Next, consider δ, equal to the radius of $\mathcal{W}$ and ω_L, ω_R to be the smallest and biggest value of $\mathcal{W}$, respectively. For such $\mathcal{W}$, δ, ω_L and ω_R consider the grid $\mathcal{S}$ in Lemma 3 with $\frac{\epsilon}{2(2N+1)\max|\hat{f}(\tilde{\omega}_j)|}$.

Clearly, it is enough to consider half-open interval $A = (a, b]$ that intersects with some $(\omega_j, \omega_{j+1}]$'s. These non-empty intersections form a sequence of half-open intervals, say $\left(u_l^L, u_l^R\right]$, $l = 1, \ldots J$, and we take $\tilde{u}_l = \tilde{\omega}_j$ from the corresponding $(\omega_j, \omega_{j+1}]$. Then, for

$$\gamma_k = \frac{\sum_{l=1}^{J}\hat{f}(\tilde{u}_l)(e^{iu_l^R s_k} - e^{iu_l^L s_k})}{is_k}$$

the following inequalities hold

$$\left(E\left|Z_f(A)-\sum_{k=-M}^{M}\gamma_k\Lambda(s_k,s_{k+1}]\right|^2\right)^{1/2}\leqslant$$

$$\leqslant\left(E\left|Z_f(A)-\sum_{j=-N}^{N}\widehat{f}(\tilde{\omega}_j)\int\check{\mathbb{I}}_{(\omega_j,\omega_{j+1}]\cap A}(s)\ d\Lambda(s)\right|^2\right)^{1/2}$$

$$+\left(E\left|\sum_{j=-N}^{N}\widehat{f}(\tilde{\omega}_j)\int\check{\mathbb{I}}_{(\omega_j,\omega_{j+1}]\cap A}(s)\ d\Lambda(s)-\sum_{k=-M}^{M}\gamma_k\Lambda(s_k,s_{k+1}]\right|^2\right)^{1/2}$$

$$\leqslant\frac{\epsilon}{2}+\sum_{l=1}^{J}|\widehat{f}(\tilde{u}_l)|\left(E\left(\int\check{\mathbb{I}}_{(u_l^L,u_l^R]}(s)d\Lambda(s)-E\sum_{k=-M}^{M}\frac{e^{iu_l^Rs_k}-e^{iu_l^Ls_k}}{is_k}\Lambda(s_k,s_{k+1}]\right)^2\right)^{1/2}$$

$$\leqslant\frac{\epsilon}{2}+\frac{\epsilon}{2}\cdot\frac{J}{2N+1}\cdot\frac{\max_l|\hat{f}(\tilde{u}_l)|}{\max_j|\hat{f}(\tilde{\omega}_j)|}\leqslant\epsilon.$$

The first inequality is a result of the triangle inequality which is also used in the second inequality together with Lemma 2. Finally the last inequality is a result of Lemma 3 and the facts that $M\leqslant 2N+1$ and $\max_l|\hat{f}(\tilde{u}_l)|\leqslant\max_j|\hat{f}(\tilde{\omega}_j)|$. This completes the proof. □

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